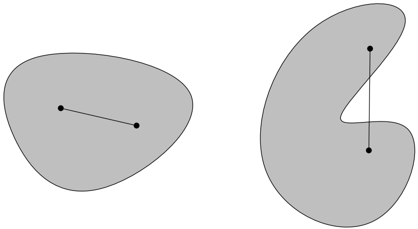
# PART 3

# CONVEX OPTIMIZATION

# Convex sets and functions

## Convex set

A set $C \subseteq \mathbb{R}^n$ is convex if for each $x, y \in C$ and $0 \leqslant \lambda \leqslant 1$, one has $\lambda x + (1-\lambda) y \in C$.



## Example

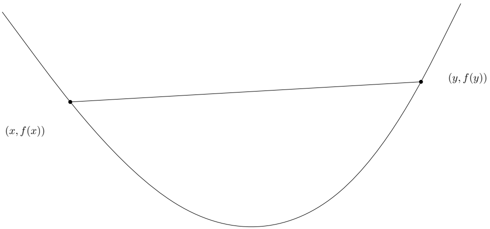
Set on the left is convex, on the right non-convex

# Convex functions

## Convex function

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is convex function, if domain of $f$ is convex and for each $x, y \in \text{domain}(f)$ and $0 \leqslant \lambda \leqslant 1$ one has

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda) f(y)$$




## Example

$\|\cdot\|$ (any norm) is a convex function, since $\|\alpha \cdot x\| = |\alpha| \cdot \|x\|$ and $\|x+y\| \leqslant \|x\| + \|y\|$. Thus $\|\lambda x + (1-\lambda)y\| \leqslant \lambda\|x\| + (1-\lambda)\|y\|$.

# Some operations preserving convexity

**Lemma 3.1**

*If $f_1, \ldots, f_n : \mathbb{R}^n \longrightarrow \mathbb{R}$ are convex functions over the same domain and $w_1, \ldots, w_n \geqslant 0$ are non-negative weights, then $\sum_{i=1}^n w_i f_i$ is convex function.*

**Lemma 3.2**

*If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is convex $A \in \mathbb{R}^{n\times m}$ and $b \in \mathbb{R}^n$, then $g : \mathbb{R}^m \longrightarrow \mathbb{R}$ with $g(x) = f(Ax + b)$ is convex.*

## Reminder

A symmetric matrix $Q \in \mathbb{R}^{n \times n}$ is called positive semi-definite if $x^T Qx \geqslant 0$ for each $x \in \mathbb{R}^n$.

## Theorem 3.3

*Let $Q \in \mathbb{R}^{n \times n}$ be a symmetric matrix. The following are equivalent.*

i) *Q is positive definite.*

ii) *All Eigenvalues of Q real and non-negative*

iii) $Q = U^T \mathit{diag}(\lambda_1, \ldots, \lambda_n) U$, *where* $U \in \mathbb{R}^{n \times n}$ *is an orthogonal matrix and* $\lambda_i \in \mathbb{R}_{\geqslant 0}$ *for* $i = 1, \ldots, n$.

## Lemma 3.4

*Let $Q \in \mathbb{R}^{n \times n}$ be symmetric and positive semidefinite, then $f(x) = x^T Qx$ is convex.*

# Sublevel sets

### Definition $C_\alpha$

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$ convex and $\alpha \in \mathbb{R}$, $C_\alpha = \{x \in \mathbb{R}^n : f(x) \leqslant \alpha\}$ is $\alpha$-sublevel set of $f$.

### Lemma 3.5

*If $f$ is convex, then $C_\alpha$ is a convex set for each $\alpha \in \mathbb{R}$.*

### Epigraph

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$ convex, $\mathbf{epi}(f) = \{(x,t) : x \in \mathrm{domain}(f), f(x) \leqslant t\}$ is epigraph of $f$.

### Lemma 3.6

*$f$ is convex if and only if $\mathbf{epi}(f)$ is convex set.*

# Convex optimization problem

## Convex optimization problem

A convex optimization problem is of the form

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leqslant b_i \quad \text{for } i = 1, \ldots, m, \end{array}$$

where $f_i$, $i = 0, \ldots, m$ are convex functions.

## Corollary 3.7

*$Q \in \mathbb{R}^{n \times n}$ positive semidefinite, $c \in \mathbb{R}^n$ $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Convex quadratic program*

$$\begin{array}{rcl} & \min x^T Q x + c^T x & \\ Ax & = & b \\ x & \geqslant & 0, \end{array}$$

*is convex optimization problem.*

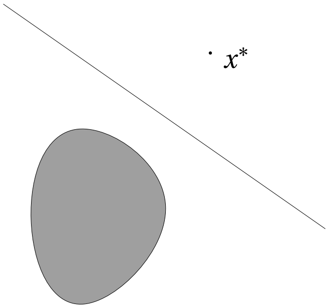
## Binary search for minimum

- Search smallest $\beta \in \mathbb{R}$ such that convex set $C_\beta = \{x \in \mathbb{R}^n : f_0(x) \leqslant \beta, f_1(x) \leqslant b_1, \ldots, f_m(x) \leqslant b_m\}$ is non-empty.
- Keep upper bound $U$ and lower bound $L$
- **Test**: Whether $C_{(L+U)/2} = \emptyset$. If yes, then $L := (L+U)/2$. If no, then $U := (L+U)/2$.
- After $O(\log((U-L)/\varepsilon)$ many steps, one obtains a value of distance $\leqslant \varepsilon$ from the optimum value.

# Separating hyperplane

**Theorem 3.8**

*If $S \subseteq \mathbb{R}^n$ is closed and convex and $x^* \notin S$, then there exists a hyperplane $c^T x = \delta$ such that $c^T s < \delta$ for each $s \in S$ and $c^T x^* > \delta$.*



# Balls and ellipsoids

The unit ball is the set $B = \{x \in \mathbb{R}^n \mid \|x\| \leqslant 1\}$. An ellipsoid $E(A,b)$ is the image of the unit ball under a affine map $t : \mathbb{R}^n \to \mathbb{R}^n$ with $t(x) = Ax + b$, where $A \in \mathbb{R}^{n \times n}$ is an invertible matrix and $b \in \mathbb{R}^n$ is a vector.

Clearly

$$E(A,b) = \{x \in \mathbb{R}^n \mid \|A^{-1}x - A^{-1}b\| \leqslant 1\}. \tag{14}$$

### Exercise

*Consider the mapping* $t(x) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x(1) \\ x(2) \end{pmatrix}$. *Draw the ellipsoid which is defined by* $t$. *What are the axes of the ellipsoid?*

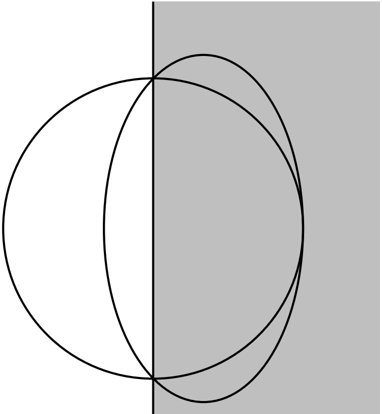
### Volume of unit ball

The volume of the unit ball is $V_n \sim \frac{1}{\pi n} \left(\frac{2e\pi}{n}\right)^{n/2}$.
Volume of ellipsoid $E(A,b)$ is equal to $|\det(A)| \cdot V_n$.

## Lemma 3.9 (Half-Ball Lemma)

*The half-ball* $H = \{x \in \mathbb{R}^n \mid \|x\| \leqslant 1, x(1) \geqslant 0\}$ *is contained in the ellipsoid*

$$E = \left\{ x \in \mathbb{R}^n \mid \left(\frac{n+1}{n}\right)^2 \left(x(1) - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} \sum_{i=2}^{n} x(i)^2 \leqslant 1 \right\} \qquad (15)$$



## Proof

Let $x$ be contained in the unit ball, i.e., $\|x\| \leqslant 1$ and suppose further that $0 \leqslant x(1)$ holds. We need to show that

$$\left(\frac{n+1}{n}\right)^2 \left(x(1) - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} \sum_{i=2}^{n} x(i)^2 \leqslant 1 \tag{16}$$

holds. Since $\sum_{i=2}^{n} x(i)^2 \leqslant 1 - x(1)^2$ holds we have

$$\begin{aligned} &\left(\frac{n+1}{n}\right)^2 \left(x(1) - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} \sum_{i=2}^{n} x(i)^2 \\ &\qquad \leqslant \left(\frac{n+1}{n}\right)^2 \left(x(1) - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2}(1 - x(1)^2) \end{aligned} \tag{17}$$

This shows that (16) holds if $x$ is contained in the half-ball and $x(1) = 0$ or $x(1) = 1$.

## Proof cont.

Now consider the right-hand-side of (17) as a function of $x(1)$, i.e., consider

$$f(x(1)) = \left(\frac{n+1}{n}\right)^2 \left(x(1) - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2}(1 - x(1)^2). \tag{18}$$

The first derivative is

$$f'(x(1)) = 2 \cdot \left(\frac{n+1}{n}\right)^2 \left(x(1) - \frac{1}{n+1}\right) - 2 \cdot \frac{n^2-1}{n^2} x(1). \tag{19}$$

We have $f'(0) < 0$ and since both $f(0) = 1$ and $f(1) = 1$, we have $f(x(1)) \leqslant 1$ for all $0 \leqslant x(1) \leqslant 1$ and the assertion follows.

## Corollary 3.10

*The half-ball* $\{x \in \mathbb{R}^n \mid x(1) \geqslant 0, \|x\| \leqslant 1\}$ *is contained in an ellipsoid* $E$*, whose volume is bounded by* $e^{-\frac{1}{2(n+1)}} \cdot V_n$.

## Ellipsoids: Convenient notation

An ellipsoid $\mathscr{E}(A,a)$ is the set
$\mathscr{E}(A,a) = \{x \in \mathbb{R}^n \mid (x-a)^T A^{-1}(x-a) \leqslant 1\}$, where $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix and $a \in \mathbb{R}^n$ is a vector.
Half-ellipsoid: $\mathscr{E}(A,a) \cap (c^T x \leqslant c^T a)$ where $c \in \mathbb{R}^n$

Proof of the correctness of next formula can be found in [**?**].

## Lemma 3.11 (Half-Ellipsoid-Theorem)

*The half-ellipsoid* $\mathscr{E}(A,b) \cap (c^T x \leqslant c^T a)$ *is contained in the ellipsoid* $\mathscr{E}'(A',a')$ *and one has* $\mathrm{vol}(\mathscr{E}')/\mathrm{vol}(\mathscr{E}) \leqslant e^{-1/(2n)}$.

## $S \subseteq R^n$ convex compact set. Suppose the following:

I) We have an ellipsoid $\mathscr{E}_{init}$ which contains $S$.

II) We have separation oracle for $S$

Ellipsoid method decides whether $\text{vol}(S) < L$ or computes a point $x^* \in S$

## Ellipsoid method

a) (Initialize): Set $\mathscr{E}(A, a) := \mathscr{E}_{init}$

b) If $\text{vol}(\mathscr{E}(A, a)) < L$, then stop.

c) If $a \in S$, then assert $S \neq \emptyset$ and stop

d) Otherwise, compute inequality $c^T x \leqslant \beta$ which is valid for $S$ and satisfies $c^T a > \beta$ and replace $\mathscr{E}(A, a)$ by $\mathscr{E}(A', a)$ computed with formula (**??**) and goto step c).

## Theorem 3.12

*The ellipsoid method computes a point in $S$ or asserts that* $\mathrm{vol}(S) < L$*.*
*The number of iterations is bounded by* $2 \cdot n \ln(\mathrm{vol}(\mathscr{E}_{init})/L)$.