Integer Points in Polyhedra Gennady Shmonin

Course description

Preliminary

General information

Semester: Spring 2009 (February 16–May 29)
Course title: Integer Points in Polyhedra (Doctoral course)
Lecturer: Dr. Gennady Shmonin (Email: gennady.shmonin@epfl.ch, Office: MA C1 553)
Credits: 4 (2h lectures + 2h exercises each week)
Grading: oral exam (+ bonus points from exercises sessions)

Schedule:
Lectures: Tuesday 14:15–16.00 (MA 10)
Exercises: Thursday 16:15–18.00 (ELG 120, changed to MA B1 524)

Organization

There are 2h of lecture and 2h of exercises every week. Exercise sessions will partly be devoted to filling the "gaps" left in the lectures (proofs of some claims, related topics and applications, questions & answers, etc.) as well as the "gaps" in background: if needed, we may briefly go through some basics of algorithms and complexity theory, theory of polyhedra, etc.

The final grade is mostly determined by the oral exam, but some extra points can be gained via exercise sessions. Exercise sheets will be issued every week on Wednesday; solutions are to be submitted on Tuesday before the lecture. Submission is not mandatory, but the points obtained for the exercises may improve your final grade. The first exercise session is supposed to be mostly a "background lecture", e.g., basics of algorithms and complexity theory.

Lecture notes and exercise sheets will be published online on the course web-page.[1] Remarks, corrections and any sort of (constructive) criticism are appreciated.

Topics

- Basics of the geometry of numbers (lattices, Hermite normal form, Blichfeldt's and Minkowski's theorems).
- Lattice basis reduction: LLL-algorithm and its applications, Lenstra's algorithm for integer linear programming in fixed dimension.
- Integer hull of a polyhedron in fixed dimension.
- Short rational generating functions for the sets of integer points in a polyhedron.

[1] http://disopt.epfl.ch/page76733.html.

- Barvinok's algorithm for counting integer points in polyhedra.
- Integer projections of polyhedra and parametric integer linear programming.

Preliminaries

- Analysis.
- Linear algebra.
- Algorithms and complexity theory.
- Linear programming and polyhedra theory.

Whilst analysis and linear algebra is rather necessary and assumed from the very first class (not only for this course, by the way, but for mathematics in general), we can devote some time (for instance, some exercise sessions) to briefly cover the topics we need from algorithms, complexity theory, linear programming, and theory of polyhedra. We do not need much, just a general understanding will be sufficient.

Textbooks

A. Barvinok, *Integer Points in Polyhedra*, *Zurich Lectures in Advanced Mathematics*, European Mathematical Society, 2008.

J. W. S Cassels, *An Introduction to the Geometry of Numbers*, vol. 99 of *Grundlehren der mathematischen Wissenschaften*, Springer, 1971.

A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, 1986.

Overview

A *polyhedron* in the Euclidean space $\mathbb{R}^n$ is a set of the form $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$, where A is an $m \times n$-matrix and b is an m-vector. The main object of our course is the set of integer points in P, i.e., $P \cap \mathbb{Z}^n$. Integer points in polyhedra, or more generally, lattice points in convex sets, is a fundamental object in two different mathematical disciplines—the geometry of numbers and integer linear programming. Despite of the common object, until recently the two were developing quite independently. The *geometry of numbers* is a branch of number theory that was originated in the 19th century by H. Minkowski and since then provided a number of fundamental results on the relations between lattice points and convex sets. *Integer linear programming* is a more recent topic in algorithms and combinatorial optimization. The main task here is to *find* an integer point in a polyhedron, if one exists.

The connexion between these disciplines was first observed by H. W. Lenstra, Jr. [6], who presented a polynomial algorithm for integer linear programming in fixed dimension.[2] The algorithm intensively exploited the techniques from the geometry of numbers. Later, the ideas of

[2] Recall that in general integer linear programming is NP-hard.

the Lenstra's algorithm gave rise to the celebrated *LLL-algorithm* (A. K. Lenstra, H. W. Lenstra, Jr., L. Lovász [7]), which found many applications in different branches of mathematics and stimulated the research of algorithms in the geometry of numbers.

A little earlier, V. N. Shevchenko [8] and A. S. Hayes and D. C. Larman [3] independently discovered that the number of vertices of the *integer hull* of P, i.e., the polyhedron $\operatorname{conv}(P \cap \mathbb{Z}^n)$, is bounded by a polynomial in the input size. Combined with the Lenstra's algorithm, this implies that the integer hull of a polyhedron in fixed dimension can be computed in polynomial time. This basically closes the study of integer linear programming in fixed dimension in its traditional statement. Indeed, after computing the integer hull of P, integer linear programming with respect to any objective $c^{\mathrm{T}} x$ is actually equivalent to linear programming. Moreover, the set of "potential" optimum solutions is essentially the set of vertices of $\operatorname{conv}(P \cap \mathbb{Z}^n)$ and they all can be listed in polynomial time.

Another significant result in this topic is due to A. Barvinok [1], who gave a polynomial algorithm for counting integer points in a polyhedron in fixed dimension. It also relies on the techniques from the geometry of numbers, but additionally uses representation of the (exponentially large even in fixed dimension) set $P \cap \mathbb{Z}^n$ as a short rational function. The algorithm was implemented and proved to be efficient also in practice. The following paper by A. Barvinok and K. Woods [2] extended the counting algorithm to the case of integer points in the *integer projections* of polyhedra in fixed dimension, i.e., sets of the form

$$\left\{x \in \mathbb{Z}^k : \exists\, y \in \mathbb{Z}^{n-k} \text{ s.th. } (x, y) \in P\right\}.$$

One of the crucial ingredients of this extension was an auxiliary result of Kannan [4, 5], who considered another generalization of integer linear programming—*parametric integer linear programming*—and gave a polynomial algorithm for this problem in fixed dimension.

In this course I intend to cover all of these remarkable results.

References

[1] A. Barvinok, A polynomial time algorithm for counting integral points in polyhedra when the dimension is fixed, *Mathematics of Operations Research* **19**(4):769–779, 1994.

[2] A. Barvinok and K. M. Woods, Short rational generating functions for lattice point problems, *Journal of the American Mathematical Society* **16**(4):957–979, 2003.

[3] A. C. Hayes and D. G. Larman, The vertices of the knapsack polytope, *Discrete Applied Mathematics* **6**(2):135–138, 1983.

[4] R. Kannan, Test sets for integer programs, ∀∃ sentences, in W. J. Cook and P. D. Seymour (eds.), *Polyhedral Combinatorics: Proceedings of a DIMACS Workshop held at the Centre for Discrete Mathematics and Theoretical Computer Science, June 12–16, 1989*, pp. 39–47, 1990.

[5] R. Kannan, Lattice translates of a polytope and the Frobenius problem, *Combinatorica* **12**(2):161–177, 1992.

[6] H. W. Lenstra, Jr., Integer programming with a fixed number of variables, *Mathematics of Operations Research* **8**(4):538–548, 1983.

[7] A. K. Lenstra, H. W. Lenstra, Jr., and L. Lovász, Factoring polynomials with rational coefficients, *Mathematische Annalen* **261**(4):515–534, 1982.

[8] V. N. Shevchenko, O chisle kraynih tochek v celochislennom programmirovanii [Russian; On the number of extreme points in integer programming], *Kibernetika* **2**:133–134, 1981.