Convexity

Prof. Friedrich Eisenbrand
Natalia Karaskova

Assignment Sheet 5

October 19, 2015

Exercise 1
The Gamma function if defined as $\Gamma(x) = \int_0^\infty r^{x-1}e^{-r}\mathrm{d}r$ for $x > 0$. Prove that

1. $\Gamma(x+1) = x\Gamma(x)$.
2. $\Gamma(n) = (n-1)!$ for positive integer n.

Exercise 2
Prove the missing step in the computation of v_n, the volume of the ball B_1^n from the lecture. Namely show that

$$\int_0^\infty nR^{n-1}e^{-\frac{R^2}{2}}\mathrm{d}R = 2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}+1\right)$$

Exercise 3 [$\star$]
Let Δ be the standard d-dimensional simplex:

$$\Delta = \{(\xi_1, \dots, \xi_{d+1}) : \xi_i \geq 0 \text{ for } i = 1, \dots, d+1 \text{ and } \xi_1 + \cdots + \xi_{d+1} = 1\}.$$

Consider Δ as a d-dimensional convex body in the affine hyperplane

$$H = \{(\xi_1, \dots, \xi_{d+1}) : \xi_1 + \cdots + \xi_{d+1} = 1\}.$$

1. Prove that the maximum volume ellipsoid E_{in} of Δ is the ball of radius $\frac{1}{\sqrt{d(d+1)}}$ centered at $r = \left(\frac{1}{d+1}, \dots, \frac{1}{d+1}\right)$.
2. Show that the constant d in John's Lemma cannot be improved. That is, show that

   $$\Delta \not\subseteq (d-\epsilon)(E_{in} - r) + r$$

   for any $\epsilon > 0$.

Exercise 4
Let V be a vector space and let $p : V \to \mathbb{R}$ be a norm. Let $K_p = \{x \in V : p(x) \leq 1\}$. Prove that K_p is a convex set, symmetric about the origin, which does not contain straight lines and such that $\cup_{\lambda \geq 0}(\lambda K_p) = V$. Conevrsely, let $K \subset V$ be a convex set, symmetric about the origin, which does not contain straight lines and such that $\cup_{\lambda \geq 0}(\lambda K) = V$. For $x \in V$ let $p_K(x) = \inf\{\lambda > 0 : x \in \lambda K\}$. Prove that p_K is a norm.

The deadline for submitting solutions is **Monday, October 26, 2015**.