Convexity

Prof. Friedrich Eisenbrand
Natalia Karaskova

Assignment Sheet 4

October 12, 2015

Exercise 1 [$\star$]
Let $\Lambda \subset \mathbb{R}^d$ be a lattice and let $v \in \Lambda \backslash \{0\}$ be a shortest non-zero vector of Λ. Show there exist a basis of Λ of the form $(v, a_2, \ldots, a_d)$.

[*Hint:* Starting from a basis $(b_1, \ldots, b_d)$ of Λ, show that there exists $U \in \mathbb{Z}^{d \times d}$ with $det(U) = \pm 1$ and $(b_1, \ldots, b_d)U = (v, a_2, \ldots, a_d)$.]

Exercise 2
Let $\Lambda \subset \mathbb{R}^d$ be a lattice, v a shortest non-zero vector in Λ and $(v, a_2, \ldots, a_d)$ a basis of Λ. Show that $pr(\Lambda)$ is a $(d-1)$-dimensional lattice with basis $(pr(a_2), \ldots, pr(a_d))$, where pr is the projection map on the subspace of $\mathbb{R}^d$ orthogonal to v, i.e.

$$\begin{aligned} pr : \mathbb{R}^d &\rightarrow \mathbb{R}^d \\ x &\mapsto \left(x - \frac{x^\intercal v}{v^\intercal v} v \right) \end{aligned}$$

Exercise 3
Let $\Lambda \subset \mathbb{R}^2$ be a lattice. Prove directly that $\mu(\Lambda)\rho(\Lambda^\star) \leq \frac{\sqrt{5}}{4}$.

Exercise 4
Draw ellipsoid E defined by $A = \begin{pmatrix} 1 & 3 \\ 2 & 9 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. What are the axes of E?

Exercise 5
Let $K \subset \mathbb{R}^d$ be a compact convex body with a non-empty interior and suppose you are given E_{in}, the ellipsoid of largest volume contained in K. Show how to compute $u \in \mathbb{Z}^d$ s.t. $\max_{x,y \in K} u^\intercal(x-y) \leq d \cdot w(K)$ by one shortest lattice vector computation, where $w(K)$ is defined to be

$$w(K) = \min_{u \in \mathbb{Z}^d \backslash \{0\}} \max_{x,y \in K} u^\intercal (x-y)$$

Exercise 6
Show how to solve the following problem in $n^{O(n)}$ calls to an oracle that computes an integer vector $u \in \mathbb{Z}^d \setminus \{0\}$ with $\max_{x,y \in K} u^T(x-y) \leq d \cdot w(K)$.

Given a convex body K, decide whether $K \cap \mathbb{Z}^d = \emptyset$.

The deadline for submitting solutions is **Monday, October 19, 2015**.