Convexity

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Assignment Sheet 10

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Definition

$C \subseteq \mathbb{R}^n$ is a *convex cone* if $\forall a, b \in C$ and $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}$, $\lambda_1 a + \lambda_2 b \in C$

$C \subseteq \mathbb{R}^n$ is a *polyhedral cone* if $C = \{x \in \mathbb{R}^n : Ax \leq 0\}$ for some $A \in \mathbb{R}^{m \times n}$.

$C \subseteq \mathbb{R}^n$ is *finitely generated* if $C = \{\sum_{i=1}^k \lambda_i d_i : \lambda \geq 0\}$ for some $k \in \mathbb{N}$ and $d_1 \ldots, d_k \in \mathbb{R}^n$

Exercise 1

Show that if $C \subseteq \mathbb{R}^n$ is a polyhedral cone, then C is a convex cone. Show that if $C \subseteq \mathbb{R}^n$ is finitely generated, then C is a convex cone.

Exercise 2

Show that if $C \subset \mathbb{R}^n$ is a convex cone, and $a^\intercal x \leq \beta$ is a valid inequality for C, such that $C \cap \{x \in \mathbb{R}^n : a^\intercal x = \beta\} \neq \emptyset$, then $\beta = 0$.

Exercise 3 [$\star$]

Let $C \subset \mathbb{R}^n$. Show that C is a polyhedral cone if and only if C is finitely generated.

[*Hint:* To show a finitely generated set is a polyhedral cone, apply the idea of Fourier-Motzkin elimination that we saw during the lecture. To obtain a finite generating set for a polyhedral cone C, consider the intersection of C with the cube $D = \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$]

Exercise 4

Let $v_1, \ldots, v_k \in \mathbb{R}^n$ be fixed and let $P = \{y \in \mathbb{R}^n : y = \sum_{i=1}^k \lambda_i v_i \text{ for some } \lambda \geq 0, \sum_{i=1}^k \lambda_i = 1\}$. Describe a real matrix C and a vector d such that

$$P = \left\{ y \in \mathbb{R}^n : \exists \lambda \in \mathbb{R}^k \text{ with } C \begin{pmatrix} y \\ \lambda \end{pmatrix} \leq d \right\}$$

Exercise 5

Let S be a polyhedron of the form $S = \left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} \in \mathbb{R}^{n+k} : x \in \mathbb{R}^n, \lambda \in \mathbb{R}^k : C \begin{pmatrix} x \\ \lambda \end{pmatrix} \leq d \right\}$ for some $C \in \mathbb{R}^{m \times (n+k)}$ and $d \in \mathbb{R}^m$. Set $S_0 = S$ and for $i = 1, \ldots, k$ define recursively

$$S_i = \Pi_{(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_{k-i})^\intercal}(S_{i-1})$$

Show that $\Pi_x(S) = S_k$ and conclude that $\Pi_x(S)$ is a polyhedron.

The deadline for submitting solutions is **Monday, December 7, 2015**