Convexity

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Assignment Sheet 1

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Exercise 1
Let $D \subseteq \mathbb{R}^d$ be a convex set and let $f : D \to \mathbb{R}$ be convex. That is, $\forall a, b \in D$, $\forall \lambda \in [0, 1]$

$$f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda) f(b)$$

a) Show that the following set is convex:

$$C = \{(x, y) : x \in D, y \geq f(x)\}$$

b) Prove or give a counterexample to the following assertion:
If $D \subseteq R^d$ is convex, $f : D \to \mathbb{R}$ and C as constructed above is a convex set, then f is convex.

Exercise 2
Recall the *Separation theorem*:
Let $C \subseteq \mathbb{R}^d$ be closed and convex. If $x^\star \notin C$, then there exists a hyperplane $a^\intercal x = \beta$ s.t. $a^\intercal x^\star < \beta$ and $\forall x \in C$ it holds that $a^\intercal x > \beta$.

In the lecture, we proved the theorem for bounded C. Extend this proof for general, unbounded C.

Exercise 3
Give a proof of *Caratheodory's theorem*:
Let $X \subseteq \mathbb{R}^d$. Then each point in $conv(X)$ is in $conv(S)$ for some $S \subseteq X$, $|S| \leq d + 1$.

Exercise 4 [$\star$]
Let $X \subseteq \mathbb{R}^2$. For each point $x \in X$, let us denote $V(x)$ the set of all points $y \in X$ that can "see" x, i.e. points s.t. the segment xy is contained in X. More formally, for $x \in X$ let

$$V(x) = \{y \in X : \forall \lambda \in [0, 1], \lambda x + (1 - \lambda)y \in X\}$$

The *kernel* of X is the set of all points $x \in X$ for which $V(x) = X$.

a) Prove that the kernel of any set $X \subseteq \mathbb{R}^2$ is convex.

b) Construct a nonempty set $X \subseteq \mathbb{R}^2$ such that each of its finite subsets can be seen from some point of X but the kernel of X is empty.

The deadline for submitting solutions is **Friday, September 25, 2015**.