Prof. Friedrich Eisenbrand
Dr. Yuri Faenza

Discussions from: December 17, 2013

Combinatorial Optimization

Fall 2013

Assignment Sheet 7

Due to the upcoming end of the semester, there is no ⋆ exercise in this sheet.

Exercise 1
Show that a subset of $\mathbb{R}^n$ is a full-dimensional ellipsoid if and only if it can be written as $\{x \in \mathbb{R}^n : ||A^{-1}(x-b)|| \leq 1\}$.

Exercise 2
In the lecture we showed that

$$E = \{x \in \mathbb{R}^n : (\frac{n+1}{n})^2 (x_1 - \frac{1}{n+1})^2 + \frac{n^2-1}{n^2} \sum_{i=2}^{n} x_i^2 \leq 1\}$$

is an ellipsoid. In order to be able to use it for the ellipsoid method, we must verify it also contains the half-ball $\{x \in \mathbb{R}^n : ||x|| \leq 1, x_1 \geq 0\}$ (this was not explicitly done in the lecture). Prove it.

Exercise 3
Show the following lemma, left unproved in class: Let P be a full dimensional polytope, F a facet of P, and suppose that all points of F satisfy $ax = \beta$. Then $F = P \cap \{x : ax = \beta\}$.

Exercise 4
A *perfect matching* is a matching of a graph that takes all vertices. Show that the maximum weighted matching problem and the maximum weighted perfect matching problem are polynomially equivalent (i.e. you can solve one of them in polynomial time if and only if you can solve the other in polynomial time).

Exercise 5
Provide an efficient algorithm that, given a polytope P defined by a polynomial number of inequalities and a point of P, finds a vertex of P.

Exercise 6

In class we showed that the matching polytope of a graph $G(V,E)$ is described by

$$P = \{x \in \mathbb{R}^E :$$

$$x_e \geq 0 \qquad \text{for } e \in E \tag{1}$$

$$x((\delta(v)) \leq 1 \qquad \text{for } v \in V \tag{2}$$

$$\textstyle\sum_{e\in E(S)} x_e \leq (|S|-1)/2 \qquad \text{for } S \subseteq V \text{ with } |S| \text{ odd} \tag{3}$$

and those are an exponential number. But we did not prove that all of them are needed. It may in fact be that our system is redundant, that is, we can define the polytope with a smaller number of inequalities (and this is the case), even to a polynomial number (but this is not the case). In this exercise, we will see a way to prove the statements in the two parenthesis above.

(i) First, we will show the validity of three methods to prove that an inequality (does not) define a facet of P (this methods extends to any full-dimensional polytopes and, with some care, also to non-full dimensional ones):

   (a) $cx \leq d$ is a facet of P if and only if there exists a point $\bar{x}$ such that $c\bar{x} > d$, while $\bar{x}$ verify all other inequalities of P;

   (b) $cx \leq d$ is a facet of P if and only if there exists a set $\{x^1, \ldots, x^{|E|}\}$ of affinely independent points of P such that $cx^1 = cx^2 = \cdots = d$.

   (c) $cx \leq d$ is a facet of P if it induces of face[1] of P that is contained in no other face of P.

(ii) Apply (i) to deduce that all inequalities from (1) define a facet.

(iii) Apply (i) to deduce that an inequality from (2) defines a facet if and only if v has degree at least 3 in G.

(iv) Apply (i) to deduce that an inequality from (3) defines a facet if and only if the subgraph $G[S]$ of G induced by G defines a hypomatchable, 2-connected graph (*hypomatchable* means that $G[S \setminus \{v\}]$ has a perfect matching, for each $v \in S$; $2-$*connected* means that $G[S \setminus \{v\}]$ is connected, for each $v \in S$. In order to prove the "if" part, you may want to recall some results from matching theory, e.g. Tutte's theorem).

(v) Deduce from (ii)-(iv) above that there exists graphs whose matching polytope has an exponential number of facets.

[1] A *face* of a polytope is a set $F = P \cap H$ for some supporting hyperplane. A facet is a face of P of maximum dimension