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Discussions from: December 03, 2013

Combinatorial Optimization

Fall 2013

Assignment Sheet 6

Exercises marked with a $\star$ can be handed in for bonus points. Due date is December 17.

In this exercise sheet, we will mostly deal with statements left unproved in the last two lectures. $f : 2^U \to \mathbb{R}_+$ is always submodular. Recall that, for a set $S \subseteq U$ and for an element $u \in U$, we have $f_u(S) := f(S \cup u) - f(S)$ (i.e. the *marginal contribution* of u to S).

Exercise 1
Let $H \subseteq U$. Prove that $g(A) = f(A \cup H)$ for $A \subseteq U$ is a submodular function.

Greedy

Input: a monotone, nonnegative submodular function $f : 2^U \to \mathbb{R}_+$, an integer k
Output: a set $S \subseteq U$ of cardinality k.

1. Let $S_0 = \emptyset$.
2. **For** $i = 1, \ldots, k$: let $u_i \in U$ such that $f_{u_i}(S_{i-1})$ is maximum. Set $S_i = S_{i-1} \cup \{u_i\}$.
3. Output S.

Exercise 2
Let S^* be the subset of U of maximum cost among those of cardinality k.
(i) show that wrt the greedy algorithm above, $f(S^*) - f(S_i) \leq (1 - 1/k)^i f(S^*)$ for each i. (Hint: find a lower bound on $f_{u_i}(S_{i-1})$, then prove the statement by induction on i.)
(ii) apply (i) to deduce that $f(S) \geq (1 - 1/e) f(S^*)$ (again wrt the greedy algorithm).

Exercise 3
Recall the following result $(*)$ proved in class, and prove its generalization $(**)$.

$(*)$ *For $A \subseteq U$, denote by $A(p)$ a random subset of A where each element appears with probability p. Then $E(f(A(p))) \geq (1-p) f(\emptyset) + p f(A)$.*

$(**)$ *Let $A, B \subseteq U$, and let $A(p)$, $B(q)$ be their independently sampled subsets, where each element of A appears in $A(p)$ with probability p and each element of B appears in $B(q)$ with probability q. Then $E(f(A(p) \cup B(q))) \geq (1-p)(1-q) f(\emptyset) + p(1-q) f(A) + q(1-p) f(B) + pq f(A \cup B)$.* (Hint: first use $(*)$ (conditioning on the outcome of $A(p)$) and Exercise 1 to show

$E(f(A(p) \cup B(q))) \geq E((1-q)f(A(p)) + qf(A(p) \cup B))$. Then apply similar arguments to conclude.)

Exercise 4 ($\star$)
Suppose f is nonnegative. For $A \subseteq U$, denote by $A(p)$ a random subset of A where each element appears with probability *at most* p (nb: different elements may have different probabilities). Show that $E(f(A(p))) \geq (1-p)f(\emptyset)$.

Exercise 5
Recall the following algorithm, where we are assuming (wlog) that $2k \leq n$, and that there is a subsets of U of $2k$ *dummy* elements whose marginal contribution to any set is 0.

Randomized Greedy

Input: a nonnegative submodular function $f : 2^U \to \mathbb{R}_+$, an integer k
Output: a set $S \subseteq U$ of cardinality k.

1. Let $S_0 = \emptyset$.
2. **For** $i = 1, \dots, k$,
   a) Let $M_i \subseteq U \setminus S_i$ be a set of size k maximizing $\sum_{u \in M_i} f_u(S_i)$.
   b) Pick an element u of M_i uniformly at random, and set $S_i = S_{i-1} \cup \{u\}$.
3. Output S_k.

In class we saw that it provides a $1-1/e$ approximation for the problem $\max\{f(S) : |S| \leq k\}$ when f is monotone. We are now going to show it gives a $1/e$ approximation for f non-monotone. Let S^* be the optimum solution. First, recall that in class we argued (for the monotone case, but the proof still holds) that: $E(f_{u_i}(S_{i-1})) \geq \frac{1}{k}E(f(S_{i-1} \cup S^*) - f(S_{i-1}))$.

(i) Prove that $f(S_{i-1} \cup S^*) \geq (1-1/k)^i f(S^*)$. (Hint: use Exercises 1 and 4)

(ii) Prove that $E(f(S_i)) \geq (1/k)(1-1/k)^{i-1} f(S^*)$, and conclude $E(f(S_k)) \geq f(S^*)/e$.

Exercise 6
Prove the following statement. Let $(M, \mathscr{I})$ be a matroid, and A, B be two basis. Then there exists $g : A \leftrightarrow B$ such that, for each $u \in A$, $(A \setminus \{u\}) \cup g(u)) \in \mathscr{I}$.

Exercise 7 ($\star$)
In class we saw that, if we pick a random set where each element appears with probability $1/2$, we obtain an expected $1/4$ approximation to the problem of computing the set A that maximizes $f(A)$. Call the output of this algorithm S, and the optimum solution S^*.

(i) Is 1/4 the right answer? Give an example where the ratio $E(f(S))/f(S^*)$ is as small as you can.

(ii) Prove that, when f is the cut function of an undirected graph, S is a 1/2 approximation in expectation.