Prof. Friedrich Eisenbrand
Dr. Yuri Faenza

Discussions from: November 5, 2013

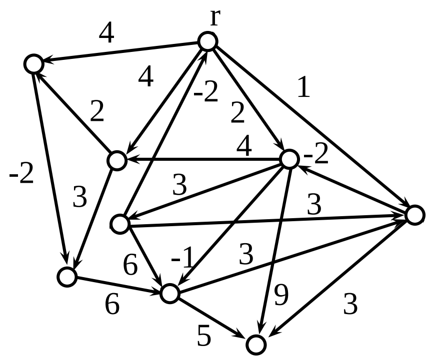
# Combinatorial Optimization

Fall 2013

Assignment Sheet 4

Exercises marked with a ⋆ can be handed in for bonus points. Due date is November 19.

**Exercise 1**
Apply the algorithm for computing a mimimum-cost $r$-arborescence to the digraph below. Then replace the edge of cost 1 with an edge of cost 5, and solve again.




**Exercise 2**
Consider the *minimum disconnection problem*: given a graph $G(V,E)$ with costs $c : E \to \mathbb{R}$, find the set $F \subseteq E$ of minimum cost such that $G(V, E \setminus F)$ is disconnected. Under which conditions this is equivalent to the minimum cut problem on $G, c$?

**Exercise 3 (⋆)**
Consider a finite set of points $V \subseteq \mathbb{R}^2$ not lying on the same line. Those induce a complete graph $G$ with vertex set $V$ and edge cost $c(uv) = ||u - v||_2$ for each $u, v \in V$. The *Voronoï diagram* of $V$ is the family of sets

$$P_v = \{x \in \mathbb{R}^2 : ||x - v||_2 = \min_{u \in V} ||x - u||_2\}$$

for $v \in V$. The *Delaunay triangulation* of $V$ is the graph $G'(V, E')$ with vertex set $V$ and edges $uv$ for each $u \neq v \in V$ such that $|P_v \cap P_u| > 1$.

(a) Show that there exists a minimum spanning tree (mst) of $G$ whose edge set is contained in $E'$. Is the previous true for every mst of $G$? Prove or give a counterexample.

(b) Suppose now to be given the adiacency lists and the incidence matrix of $G'$, as well as the cost $c$. How fast can a minimum spanning tree in $G$ be computed?

(c) Now suppose that $c$ is not the euclidean distance, but any given norm. Does any of the two statement from (a) hold? Prove or give counterexamples.

**Exercise 4**
In class we saw that a *submodular function* is a function $f : 2^I \to \mathbb{R}$ where $I$ is a ground set, that satisfies

1. For every $S, T \subseteq I$, $f(S) + f(T) \geq f(S \cap T) + f(S \cup T)$.

(a) Show that each of the following two properties is equivalent to 1.

   2. For each $S \subseteq T \subseteq I$ and $x \in I \setminus T$, we have $f(T \cup \{x\}) - f(T) \leq f(S \cup \{x\}) - f(S)$.

   3. For each $S \subseteq I$ and $x \neq y \in I \setminus S$, we have $f(S \cup \{x\}) + f(S \cup \{y\}) \geq f(S \cup \{x, y\}) + f(S)$.

(b) Consider the following modification of property 2:

   2'. For each $S \subseteq T \subseteq I$ and $x \in I$, we have $f(T \cup \{x\}) - f(T) \leq f(S \cup \{x\}) - f(S)$.

   Show how to modify 1 to $1'$ as to have equivalence between $1'$ and $2'$.

**Exercise 5**
A function is *supermodular* if $-f$ is submodular. For each of the following functions, detect if it is submodular, supermodular, both, or none of the two.
*(a)* Given a graph $G(V, E)$ with costs $c$ on the edges, let $f(S) = c(\delta(S))$ for $S \subseteq V$.
*(b)* Given a directed graph $D(V, A)$ with nonnegative costs $c$ on the arcs, let $f(S) = c(\delta_+(S))$ for $S \subseteq V$.
*(c)* Given a graph $G(V, E)$, let $f(F)$ be the number of connected components of the graph $G(V, F)$, for all $F \subseteq E$.
*(d)* Given a graph $G(V, E)$, let $f(S)$ be the number of connected components of the the subgraph of $G$ induced by $S$ (i.e. $G'(S, E')$ with $uv \in E'$ iff $uv \in E$ and $u, v \in S$), for all $S \subseteq V$.
*(e)* Given a bipartite graph $G(A \cup B, E)$, let $f(X) = |N(X)|$ for all $X \subseteq A$.
*(f)* The rank function of a matroid.
*(g)* Let $X^1, \ldots, X^n$ be discrete random variables. For $S \subseteq \{1, \ldots, n\}$, let $f(S)$ be the entropy of the random variables $\{X^i : i \in S\}$[1].

**Exercise 6**
Deduce from the random contraction min-cut algorithm an upper bound to the number of minimum cuts in a graph. Is this bound tight?

[1]The *entropy* of a random variable $X$ with possible values $x_1, \ldots, x_k$ is $H(X) = -\sum_{i=1}^{k} p(x_i) \log(p(x_i))$, where $p(x_i)$ is the probability mass function of $x_i$. The *entropy* of a set of random variables $X^1, \ldots, X^m$, where for $i =, \ldots, n$ the possible values of $X^i$ are $x^i_1, \ldots, x^i_{k_i}$ is $H(X) = -\sum_{i_1, i_2, \ldots, i_m} p(x^1_{i_1}, x^2_{i_2}, \ldots, x^m_{i_m}) \log(p(x^1_{i_1}, x^2_{i_2} \ldots, x^m_{i_m}))$, where $p$ is the joint probability distribution of $X^1, \ldots, X^m$.