Combinatorial Optimization

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Sheet 2 **October 4, 2011**

General remark:
In order to obtain a bonus for the final grading, you may hand in written solutions to the exercise marked with a star at the beginning of the exercise session on October 18.

Exercise 1

Let P be a rational polyhedron in $\mathbb{R}^n$. Show that if P equals the convex hull of its integral points, i.e. $P = \text{conv}(P \cap \mathbb{Z}^n)$, then P is integral.

Exercise 2

Consider the two following systems:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The two systems clearly define the same polyhedron. Show that the first one is TDI, but the second is not.

Exercise 3

Show the following:
If $Ax \leq b$ is TDI and $ax \leq \beta$ is a valid inequality for $\{x\colon Ax \leq b\}$, then the system $Ax \leq b, ax \leq \beta$ is also TDI.

Hint: Use duality!

Exercise 4 ($\star$)

Consider a family $S_1, \dots, S_m$ of subsets $\{1, \dots, n\}$. The *set-covering problem* is to choose a smallest number of these sets whose union is $\{1, \dots, n\}$. This is modeled in the following integer linear program

$$\begin{aligned} &\min \textstyle\sum_{j=1}^m x_j \\ &A\,x \geq \mathbf{1} \\ &x \geq 0, \text{ integral} \end{aligned}$$

where $A \in \{0,1\}^{n \times m}$ is the matrix $A(i,j) = 1$ if $i \in S_j$ and $A(i,j) = 0$ otherwise.

The goal of this exercise is to see that things are not as nice here as in the case of maximum weight matchings. We cannot expect to prove optimality of an integral solution by providing an optimal dual solution.

(i) Provide an example where the linear program (integrality-constrained ignored) has a strictly smaller solution than the integer program.

(ii) Let x^* be an optimal solution to the linear program. We are now selecting some sets in $S_1, \dots, S_m$ at random: If $x_j^* \geq 1$, then select set S_j. Otherwise select S_j with probability x_j^*. What is the expected number of selected sets?

(iii) Show that the probability that a particular element $i \in \{1, \dots, n\}$ is not covered is bounded by $(1 - 1/m)^m \leq e^{-1}$.

(iv) If this complete rounding procedure is repeated k-times, then the probability that i is not covered in any round is bounded by e^{-k}.

(v) If OPT denotes the optimum value of the integer linear program and OPT_f denotes the optimum value of the linear program, then conclude that $OPT/OPT_f = O(\log n)$.

(vi) Can you find a class of set-covering problems with $OPT/OPT_f = \Omega(\log n)$?