École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

May 24, 2018

Discrete Optimization (Spring 2018)

Assignment 12

Problem 4 can be **submitted** until June 1, 12:00 noon, into the box in front of MA C1 563. You are allowed to submit your solutions in groups of at most three students.

Problem 1

Given a graph $G = (V, E)$, a subset $S \subseteq V$ is called *stable* (or *independent*) if $|e \cap S| \leq 1$ for each $e \in E$. Assuming that G is bipartite, show that one can find a stable set of maximum cardinality in polynomial time.

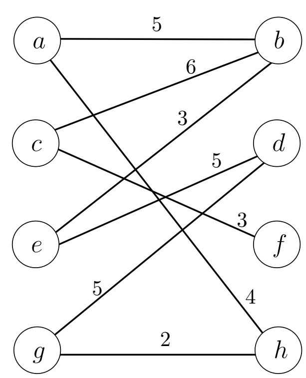
Hint: Consider the family of stable sets in a bipartite graph $G = (V, E)$, and let P be the convex hull of the corresponding indicator vectors. Describe P as $\{x \in \mathbb{R}^{|V|} : \ Ax \leq \mathbf{1}, \ x \geq 0\}$, where $A \in \mathbb{Z}^{m \times |V|}$ is totally unimodular and m is polynomial in $|V|$.

Problem 2

Given the weighted graph on the right, find the following:

a) A matching that is not perfect and has weight 15.

b) A w-vertex cover of weight 16 where at least 7 vertices have non-zero weights.



Problem 3

Consider a graph $G = (V, E)$. A matching $M \subseteq E$ is said to be *maximal* if there is no edge $e \in E \setminus M$ such that $M \cup e$ is a matching. Denote with M^* a maximum cardinality matching in G.

a) Show that $|M| \geq \frac{|M^*|}{2}$ for any maximal matching M in G.

b) Provide a graph containing a maximal matching M with $|M| = \frac{|M^*|}{2}$.

Problem 4 ($\star$)

A 2-matching in a graph is a collection of disjoint cycles that covers all the vertices. Show that a 2-matching can be computed in polynomial time, if such one exists. Note that it is allowed to pick an edge twice in a 2-matching, i.e., one can have a 2-cycle.

Hint: One may reduce the problem to finding a perfect matching in a bipartite graph.

Problem 5

Prove Hall's theorem: Let $G = (A \cup B, E)$ be a bipartite graph, and for each $S \subseteq A$, let

$$N(S) = \{v \in B : \ \exists u \in S \text{ such that } \{u, v\} \in E\}.$$

Then, G has a matching of size $|A|$ if and only if $|N(S)| \geq |S|$ for all $S \subseteq A$.