École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

May 18, 2018

Discrete Optimization (Spring 2018)

Assignment 11

Problem 4 could be **submitted** until May 25 12:00 noon into the right box in front of MA C1 563.You are allowed to submit your solutions in groups of at most three students.

Problem 1

Let $M \in \mathbb{Z}^{n \times m}$ be totally unimodular. Prove that the following matrices are totally unimodular as well:

1. M^T
2. $(M \;\; I_n)$
3. $(M \;\; -M)$
4. $M \cdot (I_n - 2e_j^T e_j)$ for some j.

I_n is the $n \times n$ identity matrix and e_j is the vector having a 1 in the j-th component, and 0 in the other components.

Problem 2

Let G be a graph and let A be its node-edge incidence matrix. We have seen in class that if G is bipartite then A is totally unimodular. Prove the converse, *i.e.*, if A is totally unimodular then G is bipartite.

Problem 3

Consider a bipartite graph $G = (A \cup B, E)$. Assume there exist matchings M_A and M_B covering vertices $A_1 \subseteq A$ and $B_1 \subseteq B$, respectively. Prove that there always exists a matching that covers $A_1 \cup B_1$.
Hint: The symmetric difference $M_A \Delta M_B$ consists of only cycles and paths.

Problem 4 ($\star$)

Given a graph $G(V, E)$, a perfect matching of G is a matching which covers all the vertices (equivalently, a matching of cardinality $|V|/2$). Suppose you are given an oracle that, given a graph G, tells you whether G has a perfect matching or not. Show how to use this oracle to find a maximum cardinality matching of a graph $G(V, E)$, using at most $|V| + |E|$ calls to the oracle.
Hint: you should modify the graph at each call of the oracle.

Problem 5

A family of sets $\mathcal{C} \subset 2^{[n]}$ is a chain if for all $S, T \in \mathcal{C}$ we have either $S \subseteq T$ or $T \subseteq S$. Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are two chains. Let $A \in \{0,1\}^{(|\mathcal{C}_1|+|\mathcal{C}_2|)\times n}$ be the incidence matrix of $\mathcal{C}_1 \cup \mathcal{C}_2$, i.e. $A_{S,i} = 1$ if $i \in S$ and 0 otherwise, for $i = 1, \dots, n$ and $S \in \mathcal{C}_1 \cup \mathcal{C}_2$. Prove that A is totally unimodular.
Hint: use induction on the size of a square submatrix of A.