École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

May 12, 2017

Discrete Optimization (Spring 2017)

Assignment 10

Problem 1 can be submitted until May 19 12:00 noon into the right box in front of MA C1 563.
You are allowed to submit your solutions in groups of at most three students.

Problem 1 ($\star$)

Show the following. If $P \subseteq \mathbb{R}^n$ is a bounded and full-dimensional polyhedron, then there exist vertices $v_1, \dots, v_{n+1}$ of P that are affinely independent, i.e., $v_2 - v_1, v_3 - v_1, \dots, v_{n+1} - v_1$ are linearly independent. *Hint: If $a^T x = \beta$ is some hyperplane, where $a \in \mathbb{R}^n \setminus \{0\}$, then there exists a vertex of P that is not contained in that hyperplane.*

Problem 2

Let $a_1, \dots, a_n \in \mathbb{Z}^n$ be linearly independent. Show that

$$\text{vol}(\text{conv}(0, a_1, \dots, a_n)) = |\det(a_1, \dots, a_n)|/n!.$$

Problem 3

Let $P = \{x \in \mathbb{R}^n \mid Ax \le b\}$ be a polyhedron and $\varepsilon > 0$ be a real number. Show that $P_\varepsilon = \{x \in \mathbb{R}^n \mid Ax \le b + \varepsilon \cdot \mathbf{1}\}$ is full-dimensional if $P \ne \emptyset$.

Problem 4

Let $a \in \mathbb{Q}^n$, $A \in \mathbb{Q}^{m\times n}$ and $b \in \mathbb{Q}^m$ be given as the input and $P := \{x \in \mathbb{R}^n : \; Ax \le b\}$. Show that the corresponding *separation problem* can be solved in time polynomial in m, n, and the binary encoding length of a, A and b: Determine whether $a \in P$ and if not compute an inequality $c^T x \le \beta$ which is valid for P with $c^T a > \beta$.

Problem 5

Let $P = \{x \in \mathbb{R}^n : Ax \le b\}$ be a full dimensional $0/1$ polytope and $c \in \mathbb{Z}^n$. A polytope in $\mathbb{R}^n$ is $0/1$ if the set of its vertices is a subset of $\{0, 1\}^n$. We will show how we can use the ellipsoid method to solve the optimization problem $\max\left\{c^\top x : x \in P\right\}$.

Define $z^* := \max\left\{c^\top x : x \in P\right\}$ and $c_{\max} := \max\{|c_i| : 1 \le i \le n\}$.

i) Show that the ellipsoid method requires $O(n^3 \log(n) c_{max})$ iterations to decide whether $P \cap (c^\top x \ge \beta) = \emptyset$ for some integer β. (Find a suitable initial ellipsoid and stopping value L)

ii) Show that we can use binary search to find z^* with $\log(nc_{\max})$ calls to the ellipsoid method.

iii) Show how you can find an optimal solution x^* such that $c^\top x^* = z^*$ in polynomial time.