École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

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**Discrete Optimization** (Spring 2018)

# Assignment 9

**Problem 2** can be **submitted** until May 4, 12:00 noon, into the box in front of MA C1 563. You are allowed to submit your solutions in groups of at most three students.

## Problem 1

Suppose you are given an algorithm that on input $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$ decides the feasibility of the system $Ax \leq b$, in time $\text{poly}(n, m, \log B)$, where $B = \max\{|A_{ij}|, |b_i| : \; i \in [m], j \in [n]\}$. For simplicity assume that $\text{rank}(A) = n$.

Design an algorithm that computes a basic feasible solution of $P(A, b) := \{x \in \mathbb{R}^n : \; Ax \leq b\}$ if $P(A, b)$ is feasible. The algorithm should run in time $\text{poly}(n, m, \log B)$.

*Hint: rank$(A) = n$ implies that $P(A, b)$ has vertices, and each hyperplane $H_i := \{x \in \mathbb{R}^n : \; A_i x = b_i\}$, where $A_i$ is the i-th row of A, either contains a vertex of $P$ or $P \cap H_i = \emptyset$.*

**Solution:**

Using the hint we could design the following algorithm to find a feasible basis $S$ corresponding to a vertex of $P$.

Input: $A \in \mathbb{Z}^{m\times n}, \; b \in Z^m$
Output: a feasible basis $S \subseteq [m]$ (i.e. $|S| = n$, $A_S$ is non-singular and $x^* = A_S^{-1} b_S$ is a vertex of the polytope $P$)
$S := \emptyset$
**for** $j = 1, \ldots, m$
  **if** $S \cup \{j\}$ induces lin. indep. set of rows of $A$
   **and** the linear system $Ax \leq b, \; A_k x = b_k, \; \forall k \in S \cup \{j\}$ is feasible
     $S := S \cup \{j\}$
**return** $S$

The algorithm is clearly correct: at every iteration we ensure that the rows in $S$ are linearly independent and that the system $Ax \leq b, \; A_k x = b_k, \; \forall k \in S$ is feasible, and moreover from linear algebra we know that any independent set of rows which is not maximal can be completed to a basis. This implies that the rows of the output $S$ form a feasible basis. Hence, by solving the system $A_k x = b_k, \; \forall k \in S$ we obtain a unique solution $x^*$ which is a vertex of $P$ by definition. The algorithm performs $m$ iterations, each of which takes polynomial time (in $n$, $m$ and $\log B$). Finally, finding $x^*$ can be done by using Gaussian elimination (equivalently, by computing the appropriate matrix inverse), again in time $\text{poly}(n, m, \log B)$.

## Problem 2 ($\star$)

Show the following. If $P \subseteq \mathbb{R}^n$ is a bounded and full-dimensional polyhedron, then there exist vertices $v_1, \ldots, v_{n+1}$ of $P$ that are affinely independent, i.e., $v_2 - v_1, v_3 - v_1, \ldots, v_{n+1} - v_1$ are linearly independent. *Hint: If $a^T x = \beta$ is some hyperplane, where $a \in \mathbb{R}^n \setminus \{0\}$, then there exists a vertex of $P$ that is not contained in that hyperplane.*

**Solution:**

Note that $P$ is a polytope and thus the convex hull of its vertices. We will consecutively choose vertices of $P$ while ensuring that each newly selected vertex is affinely independent to the previous ones. Initially, we can choose the first vertex arbitrarily, as every single point is affinely independent. Now, assume that we have already chosen vertices $v_1, \ldots, v_k$. Out of the remaining vertices we pick as $v_{k+1}$ an arbitrary vertex of $P$ such that $v_1, \ldots, v_k, v_{k+1}$ is affinely independent. We continue this process until $k$ is such that all the remaining vertices are affinely dependent with $v_1, \ldots, v_k$. This means that all the vertices and thus their convex hull $P$ are contained in the $k-1$-dimensional affine subspace defined by $v_1, \ldots, v_k$. Since $P$ is full-dimensional, this can only happen for $k = n+1$. At that point we are done.

**Problem 3**

Let $a_1, \ldots, a_n \in \mathbb{Z}^n$ be linearly independent. Show that

$$\text{vol}(\text{conv}(0, a_1, \ldots, a_n)) = |\det(a_1, \ldots, a_n)|/n!.$$

**Solution:**

In Problem 6 of Assignment 8 we saw that the volume of the standard simplex is $\frac{1}{n!}$. Note that the simplex spanned by $(0, a_1, \ldots, a_n)$ is the image of the standard simplex under the linear map $A = (a_1, \ldots, a_n)$. Known results from linear algebra tell us that $\text{vol}(A(K)) = |\det A| \text{vol}(K)$ for any convex body $K$ (including polytopes) and any affine map $A$. Thus the claim follows.

**Problem 4**

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a full dimensional 0/1 polytope and $c \in \mathbb{Z}^n$. A polytope in $\mathbb{R}^n$ is 0/1 if the set of its vertices is a subset of $\{0,1\}^n$. We will show how we can use the ellipsoid method to solve the optimization problem $\max\{c^\top x : x \in P\}$.

Define $z^* := \max\{c^\top x : x \in P\}$ and $c_{\max} := \max\{|c_i| : 1 \leq i \leq n\}$.

i) Show that the ellipsoid method requires $O(n^3 \log(n) c_{max})$ iterations to decide whether $P \cap (c^\top x \geq \beta) = \emptyset$ for some integer $\beta$. (Find a suitable initial ellipsoid and a stopping value $L$).

ii) Show that we can use binary search to find $z^*$ with $\log(n c_{\max})$ calls to the ellipsoid method.

iii) Show how you can find an optimal solution $x^*$ such that $c^\top x^* = z^*$ in polynomial time.

**Solution:**

i) First note that all 0/1 polytopes are contained inside the ball centered at $\frac{1}{2} \cdot \mathbf{1}$ with radius $\sqrt{n}/2$. We can upper bound the volume of this ball by $\sqrt{n}^n$. Then observe that $P \cap (c^\top x \geq \beta) = \emptyset$ iff $P' := P \cap (c^\top x \geq \beta - 1/2) = \emptyset$, since the vertices of $P$ are integral. So we want to lower bound the volume of $P'$. Let $x_0$ be an integral vertex in $P'$. Since $P$ is full dimensional it contains a simplex $\Delta = \text{conv}\{x_0, x_1, \cdots, x_n\}$ of volume $1/n!$. The idea is to scale this simplex so that it is contained in $P'$. We define $\alpha = \frac{1}{2nc_{\max}}$ and consider the simplex $\Delta' = [z_0, z_1, \cdots, z_{n-1}]$ where $z_i = x_0 + \alpha(x_i - x_0)$. To see that $\Delta'$ is contained in $P'$ we can check that each $z_i$ is in $P$ and satisfies $c^\top z_i \geq \beta$. Then we obtain that

$$\text{vol}(P') \geq \text{vol}(\Delta') \geq \frac{1}{n!}\left(\frac{1}{2nc_{\max}}\right)$$

Setting $L = \text{vol}(\Delta')$ and using the fact that the ellipsoid method terminates in $O(n \log(\text{vol}(I_{\text{init}})/L))$ gives us the correct bound.

ii) We use the fact that the value of $c^\top x$ lies between $-nc_{\max}$ and $nc_{\max}$ for any vertex $x$ of $P$ to conclude that $z^*$ must also lie within these bounds. Using binary search on this interval of integer points takes $\log(nc_{\max})$ steps.

iii) The algorithm described in $(i)$ and $(ii)$ gives us the optimal value $z^*$ but also a point $y \in P$ such that $z^* \geq c^\top y \geq z^* - 1$. We now take this point and project it onto the hyperplane $c^\top x = z^*$. Let $y'$ be the projection. If $y' \in P$ then we are done, otherwise we find a point on the line segment $yy'$ that intersects a facet of $P$. We have now reduced the dimension of our problem by one and can proceed by once again projecting this new point onto the hyperplane $c^\top x = z^*$. Continuing in this way we will arrive at an optimal solution in polynomial time.

**Problem 5**

Generalize the Half-ball lemma shown in class. Given vectors $c \in \mathbb{R}^n$ and $a \in \mathbb{R}^n$, and a symmetric positive definite matrix $A \in \mathbb{R}^{n\times n}$, provide a formula for the ellipsoid containing:

a) The half-ball $H = \{x \in \mathbb{R}^n \mid \|x\| \leq 1,\ c^T x \geq 0\}$;

b) The half-ellipsoid $\mathcal{H}(A, a) = \{x \in \mathbb{R}^n \mid (x - a)^T A^{-1}(x - a) \leq 1,\ c^T x \leq c^T a\}$.

**Solution:**

For convenience, we use a different representation of the ellipsoid than before. Here, we define it as

$$\mathcal{E}(A, a) = \{x \in \mathbb{R}^n \mid (x - a)^T A^{-1}(x - a) \leq 1\}, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a positive definite (PD) matrix, and $a \in \mathbb{R}^n$ is the center. In order to establish a relation with the ellipsoid representation from Assignment 8, we remark that for every PD matrix $A$ there exists a unique PD matrix, denote it with $A^{1/2}$, such that $A = A^{1/2}A^{1/2}$. Thus,

$$\mathcal{E}(A, a) = \{x \in \mathbb{R}^n \mid (x - a)^T \underbrace{(A^{1/2})^{-1}}_{=(A^{1/2})^{-T}}(A^{1/2})^{-1}(x - a) \leq 1\}.$$

a) Denote with $e_1 \in \mathbb{R}^n$ the unit vector with the first coordinate equal to 1, and with $I_n$ the $n \times n$ identity matrix. When rewriting the ellipsoid from the Half-ball lemma[1] in the form of (1), we obtain that the half-ball $\{x \in \mathbb{R}^n \mid \|x\| \leq 1,\ e_1^T x \geq 0\}$ is contained in the ellipsoid

$$\mathcal{E}(A(e_1), a(e_1)),\ \text{where } A(e_1) = \frac{n^2}{n^2 - 1}\left(I_n - \frac{2}{n+1}e_1e_1^T\right) \text{ and } a(e_1) = \frac{1}{n+1}e_1.$$

Without loss of generality we could assume that $\|c\| = 1$. Similarly to the proof of the Half-ball lemma in the lecture notes, one can verify that $e_1$ can be replaced with $c$ in the above formulas. I.e., the ellipsoid $\mathcal{E}(A(c), a(c))$ contains the half-ball $H$ given in the problem statement.

b) By following the initial discussion, we get that the ellipsoid $\mathcal{E}(A, a)$ represented as in (1) is the image of the unit ball $\mathcal{E}(I, 0) = \{y \in \mathbb{R}^n \mid \|y\| \leq 1\}$ under the map $t(y) = A^{1/2}y + a$. Thus, the inverse $t^{-1}(x) = (A^{1/2})^{-1}(x - a)$ maps $\mathcal{E}(A, a)$ to $\mathcal{E}(I, 0)$. The half-space $\{x \in \mathbb{R}^n \mid\ c^T x \leq c^T a\}$ is mapped by $t^{-1}(x)$ into the following half-space:

$$\{\underbrace{t^{-1}(x)}_{y} \in \mathbb{R}^n \mid\ c^T \underbrace{x}_{t(y)} \leq c^T a\} = \{y \in \mathbb{R}^n \mid\ c^T(A^{1/2}y + a) \leq c^T a\}$$

$$= \{y \in \mathbb{R}^n \mid\ \underbrace{-c^T A^{1/2}}_{(-A^{1/2}c)^T} y \geq 0\}.$$

[1] Lemma 8.1 in the lecture notes.

Furthermore, we can normalize the vector $-A^{1/2}c$ to

$$\bar{c} = \frac{-A^{1/2}c}{\|-A^{1/2}c\|} = -\frac{A^{1/2}c}{\sqrt{c^T Ac}}.$$

From part a) we know that the half-ball $\{y \in \mathbb{R}^n \mid \|y\| \leq 1,\ \bar{c}^T y \geq 0\}$ is contained in the ellipsoid $\mathcal{E}(A(\bar{c}), a(\bar{c}))$. This means that our task has been reduced to computing the image of $\mathcal{E}(A(\bar{c}), a(\bar{c}))$ under $t(y)$. The desired ellipsoid is:

$$\{t(y) \in \mathbb{R}^n \mid y \in \mathcal{E}(A(\bar{c}), a(\bar{c}))\} = \{\underbrace{t(y)}_{x} \in \mathbb{R}^n \mid (\underbrace{y}_{t^{-1}(x)} - a(\bar{c}))^T A(\bar{c})^{-1} (\underbrace{y}_{t^{-1}(x)} - a(\bar{c})) \leq 1\} \tag{2}$$

Let us unfold the expressions $a(\bar{c})$ and $A(\bar{c})$ now. We obtain that

$$t^{-1}(x) - a(\bar{c}) = (A^{1/2})^{-1}(x-a) - \frac{1}{n+1}\bar{c} = (A^{1/2})^{-1}\left(x - \left(a + \frac{1}{n+1}A^{1/2}\bar{c}\right)\right).$$

For simplicity, we introduce new vectors

$$b := -A^{1/2}\bar{c} = \frac{Ac}{\sqrt{c^T Ac}} \quad \text{and} \quad \bar{a} := a + \frac{1}{n+1}A^{1/2}\bar{c} = a - \frac{b}{n+1}.$$

Then, $(t^{-1}(x) - a(\bar{c}))^T$ can be written as $(x - \bar{a})^T (A^{1/2})^{-1}$ and the expression in (2) becomes:

$$\{x \in \mathbb{R}^n \mid (x - \bar{a})^T \underbrace{(A^{1/2})^{-1} A(\bar{c})^{-1} (A^{1/2})^{-1}}_{(A^{1/2} \cdot A(\bar{c}) \cdot A^{1/2})^{-1}} (x - \bar{a}) \leq 1\}$$

Finally, the desired ellipsoid is $\mathcal{E}(\bar{A}, \bar{a})$, where we define:

$$\begin{aligned}
\bar{A} = A^{1/2} \cdot A(\bar{c}) \cdot A^{1/2} &= A^{1/2}\left(\frac{n^2}{n^2-1}\left(I_n - \frac{2}{n+1}\bar{c}\bar{c}^T\right)\right)A^{1/2} \\
&= \frac{n^2}{n^2-1}\left(A^{1/2}A^{1/2} - \frac{2}{n+1}A^{1/2}\bar{c}\bar{c}^T A^{1/2}\right) \\
&= \frac{n^2}{n^2-1}\left(A - \frac{2}{n+1}bb^T\right).
\end{aligned}$$

For more details on the topic see Chapter 3 of Grötschel et al.[2] and references therein.

[2] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization, volume 2 of Algorithms and Combinatorics*. Springer, 1988.