École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

April 26, 2018

Discrete Optimization (Spring 2018)

Assignment 9

Problem 2 can be **submitted** until May 4, 12:00 noon, into the box in front of MA C1 563. You are allowed to submit your solutions in groups of at most three students.

Problem 1

Suppose you are given an algorithm that on input $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$ decides the feasibility of the system $Ax \leq b$, in time $\text{poly}(n, m, \log B)$, where $B = \max\{|A_{ij}|, |b_i| : \ i \in [m], j \in [n]\}$. For simplicity assume that $\text{rank}(A) = n$.

Design an algorithm that computes a basic feasible solution of $P(A, b) := \{x \in \mathbb{R}^n : \ Ax \leq b\}$ if $P(A, b)$ is feasible. The algorithm should run in time $\text{poly}(n, m, \log B)$.

Hint: rank$(A) = n$ implies that $P(A, b)$ has vertices, and each hyperplane $H_i := \{x \in \mathbb{R}^n : \ A_i x = b_i\}$, where A_i is the i-th row of A, either contains a vertex of P or $P \cap H_i = \emptyset$.

Problem 2 ($\star$)

Show the following. If $P \subseteq \mathbb{R}^n$ is a bounded and full-dimensional polyhedron, then there exist vertices $v_1, \ldots, v_{n+1}$ of P that are affinely independent, i.e., $v_2 - v_1, v_3 - v_1, \ldots, v_{n+1} - v_1$ are linearly independent. *Hint: If $a^T x = \beta$ is some hyperplane, where $a \in \mathbb{R}^n \setminus \{0\}$, then there exists a vertex of P that is not contained in that hyperplane.*

Problem 3

Let $a_1, \ldots, a_n \in \mathbb{Z}^n$ be linearly independent. Show that

$$\text{vol}(\text{conv}(0, a_1, \ldots, a_n)) = |\det(a_1, \ldots, a_n)|/n!.$$

Problem 4

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a full dimensional 0/1 polytope and $c \in \mathbb{Z}^n$. A polytope in $\mathbb{R}^n$ is 0/1 if the set of its vertices is a subset of $\{0, 1\}^n$. We will show how we can use the ellipsoid method to solve the optimization problem $\max\{c^\top x : x \in P\}$.

Define $z^* := \max\{c^\top x : x \in P\}$ and $c_{\max} := \max\{|c_i| : 1 \leq i \leq n\}$.

i) Show that the ellipsoid method requires $O(n^3 \log(n) c_{max})$ iterations to decide whether $P \cap (c^\top x \geq \beta) = \emptyset$ for some integer β. (Find a suitable initial ellipsoid and a stopping value L).

ii) Show that we can use binary search to find z^* with $\log(n c_{\max})$ calls to the ellipsoid method.

iii) Show how you can find an optimal solution x^* such that $c^\top x^* = z^*$ in polynomial time.

Problem 5

Generalize the Half-ball lemma shown in class. Given vectors $c \in \mathbb{R}^n$ and $a \in \mathbb{R}^n$, and a symmetric positive definite matrix $A \in \mathbb{R}^{n \times n}$, provide a formula for the ellipsoid containing:

a) The half-ball $H = \{x \in \mathbb{R}^n \mid \|x\| \leq 1, \ c^T x \geq 0\}$;

b) The half-ellipsoid $\mathcal{H}(A, a) = \{x \in \mathbb{R}^n \mid (x - a)^T A^{-1} (x - a) \leq 1, \ c^T x \leq c^T a\}$.