École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

May 15, 2017

Discrete Optimization (Spring 2017)

Assignment 9

Problem 2 can be submitted until **Monday**, **May 22 18:00** into the right box in front of MA C1 563.
You are allowed to submit your solutions in groups of at most three students.

Problem 1

Let $Ax \leq b$ be a system of inequalities where each component of A and b is an integer bounded by B in absolute value. Show that $Ax \leq b$ is feasible if and only if $Ax \leq b$, $-B^n \cdot n^{n/2} \cdot n \cdot B \leq x_i \leq B^n \cdot n^{n/2} \cdot n \cdot B, \ \forall i \in [n]$ is feasible.

Hint: Consider a feasible point x^* and the index sets $I = \{i : \ x_i^* \geq 0\}$ and $J = \{j : x_j^* \leq 0\}$. The polyhedron defined by $Ax \leq b, \ x_i \geq 0, \ i \in I, \ x_j \leq 0, \ j \in J$ is feasible and has vertices. Estimate the infinity norm of a vertex.

Problem 2 ($\star$)

[**updated**] Suppose that there exists an algorithm that on input $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$ decides the feasibility of the system $Ax \leq b$, in time $\text{poly}(n, m, \log B)$, where B is an upper bound on each absolute value of an entry of A and b.

i) Let the system $Ax \leq b$ be feasible. Show that there exists a polynomial time (in n, m and $\log B$) algorithm that on input A, b determines a feasible solution of $Ax \leq b$.

   Hint: Without loss of generality one can assume that $P := \{x \in \mathbb{R}^n : \ Ax \leq b\}$ is a polytope (i.e. a bounded polyhedron) since by Problem 1 one can always add the box constraint: $-B^{n+1} \cdot n^{n/2+1} \leq x_i \leq B^{n+1} \cdot n^{n/2+1}, \ \forall i \in [n]$.

   This further implies that P has vertices, and each hyperplane $H_j := \{x \in \mathbb{R}^n : \ A_j x = b_j\}$ (A_j is the j-th row of A) either contains a vertex of P or its corresponding constraint $A_j x \leq b_j$ is completely redundant (i.e. $P \cap H_j = \emptyset$).

   Argue that the algorithm below is correct, use S to obtain a vertex x^* of P and show that the total execution time is $\text{poly}(n, m, \log B)$.

   Input: $A \in \mathbb{Z}^{m \times n}, \ b \in Z^m$
   Output: a feasible basis $S \subseteq [m]$ (i.e. $|S| = n$, A_S is non-singular and $x^* = A_S^{-1} b_S$ is a vertex of the polytope P)
   $S := \emptyset$
   for $j = 1, \ldots, m$
       if $S \cup \{j\}$ induces lin. indep. set of rows of A
         and the linear system $Ax \leq b, \ A_k x = b_k, \ \forall k \in S \cup \{j\}$ is feasible
             $S := S \cup \{j\}$
   return S

ii) Let $c \in \mathbb{Z}^n$ such that $\max\{cx : Ax \leq b\} < \infty$. Using binary search, show that there exists a polynomial time (in n, m and $\log B$) algorithm that on input A, b, c determines the value of $\max\{cx : Ax \leq b\}$. Here B is an upper bound on the absolute value of each entry of A, b and c.

   Hint: As in the hint of part ii) one can assume that P is a polytope. By Problem 5(a) from Assignment 8 we know that if x_1, x_2 are vertices of P and $cx_1 \neq cx_2$, then $|cx_1 - cx_2| \geq 1/L^2$, where $L = B^n n^{n/2}$. Use binary search to find β such that $P' = P \cap \{x \in \mathbb{R}^n : \ cx \geq \beta\}$ contains *only* optimal vertices of P and modify the algorithm from part i) to obtain an optimal basis.

Problem 3

Let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix and $b \in \mathbb{R}^n$ a vector. The ellipsoid $E(A, b)$ is defined as the image of the unit ball under the linear mapping $t(x) = Ax + b$. Show that

$$E(A, b) = \left\{ x \in \mathbb{R}^n : (x - b)^\top A^{-\top} A^{-1} (x - b) \leq 1 \right\}$$

Problem 4

Draw $E(A, b)$ for $A = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Problem 5

Show that the unit simplex $\Delta = \text{conv}\{0, e_1, \cdots, e_n\} \subset \mathbb{R}^n$ has volume $\dfrac{1}{n!}$.

Problem 6

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a full dimensional 0/1 polytope and $c \in \mathbb{Z}^n$. We will show how we can use the ellipsoid method to solve the optimization problem $\max \left\{ c^\top x : x \in P \right\}$.

Define $z^* := \max \left\{ c^\top x : x \in P \right\}$ and $c_{\text{max}} := \max \{|c_i| : 1 \leq i \leq n\}$.

i) Show that the ellipsoid method requires $O(n^3 \log(n) c_{max})$ iterations to decide whether $P \cap (c^\top x \geq \beta) = \emptyset$ for some integer β. (Find a suitable initial ellipsoid and stopping value L)

ii) Show that we can use binary search to find z^* with $\log(n c_{\text{max}})$ calls to the ellipsoid method.

iii) Show how you can find an optimal solution x^* such that $c^\top x^* = z^*$ in polynomial time.