École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

April 19, 2018

Discrete Optimization (Spring 2018)

Assignment 8

Problem 1 can be **submitted** until April 27, 12:00 noon, into the box in front of MA C1 563. You are allowed to submit your solutions in groups of at most three students.

Problem 1 ($\star$)

Consider the following problem. We are given $B \in \mathbb{N}$, and a set of integer points

$$S = \{p \in \mathbb{Z}^n : \ 0 \leq p_i \leq B, \ \forall i = 1, \ldots, n\},$$

whose points are all colored blue but one, which is red. We have an oracle that, given $i \in \{1, \ldots, n\}$ and $\alpha \in \{0, \ldots, B\}$, tells us whether there exists a red point $x^* \in S$ with $x_i^* \leq \alpha$. Give an algorithm to find the red point using $O(n \log(B))$ many oracle calls.

Problem 2

Let $P := \{x \in \mathbb{R}^n : \ Ax = b, \, x \geq 0\}$ be a polyhedron and $\min\{cx : x \in P\}$ be the corresponding primal linear program. Assume that all the coefficients of A, b and c are integral and bounded in absolute value by given $B \in N$, and furthermore let $L := B^n n^{n/2}$.

(a) Show the following: If x_1, x_2 are vertices of P and $cx_1 \neq cx_2$, then $|cx_1 - cx_2| \geq 1/L^2$.

(b) Let x^* and y^* be feasible solutions of the primal and dual linear program respectively. Conclude the following from the above: If $|cx^* - by^*| < 1/L^2$, then each vertex x of P with $cx \leq cx^*$ is an optimal solution of the primal.

Problem 3

Let $Ax \leq b$ be a system of inequalities where each component of A and b is an integer bounded by B in absolute value. Show that $Ax \leq b$ is feasible if and only if $Ax \leq b$, $-B^n \cdot n^{n/2} \cdot n \cdot B \leq x_i \leq B^n \cdot n^{n/2} \cdot n \cdot B, \ \forall i \in [n]$ is feasible.

Hint: Consider a feasible point x^* and the index sets $I = \{i : \ x_i^* \geq 0\}$ and $J = \{j : x_j^* \leq 0\}$. The polyhedron defined by $Ax \leq b, \ x_i \geq 0, \ i \in I, \ x_j \leq 0, \ j \in J$ is feasible and has vertices. Estimate the infinity norm of a vertex.

Problem 4

Let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix and $b \in \mathbb{R}^n$ a vector. The ellipsoid $E(A, b)$ is defined as the image of the unit ball under the linear mapping $t(x) = Ax + b$. Show that

$$E(A, b) = \{x \in \mathbb{R}^n : (x - b)^\top A^{-\top} A^{-1}(x - b) \leq 1\}.$$

Problem 5

Draw $E(A, b)$ for $A = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Problem 6

Show that the unit simplex $\Delta = \mathrm{conv}\{0, e_1, \cdots, e_n\} \subset \mathbb{R}^n$ has volume $\dfrac{1}{n!}$.