**Discrete Optimization** (Spring 2018)

# Assignment 7

**Problem 3** can be **submitted** until April 20 12:00 noon into the box in front of MA C1 563. You are allowed to submit your solutions in groups of at most three students.

### Problem 1

Determine the dual program for the following linear program:

$$\begin{array}{rrcr} \min & 3x_1 + 2x_2 - 3x_3 + 4x_4 & & \\ & 2x_1 - 2x_2 + 3x_3 + 4x_4 & \leq & 3 \\ & x_2 + 3x_3 + 4x_4 & \geq & -5 \\ & 2x_1 - 3x_2 - 7x_3 - 4x_4 & = & 2 \\ & x_1 \geq 0 & & \\ & x_4 \leq 0 & & \end{array}$$

**Solution:**
After some simplifications, one obtains:

$$\begin{array}{rrcr} \max & 3y_1 - 5y_2 + 2y_3 & & \\ & 2y_1 + 2y_3 & \leq & 3 \\ & -2y_1 + y_2 - 3y_3 & = & 2 \\ & 3y_1 + 3y_2 - 7y_3 & = & -3 \\ & y_1 + y_2 - y_3 & \geq & 1 \\ & y_1 \leq 0 & & \\ & y_2 \geq 0 & & \end{array}$$

### Problem 2

In the GitHub repository of the course, in the 'Programming' folder, you will find the file 'Simplex.py' which contains the code to run the simplex algorithm on non-degenerate linear programs in standard form. However, some code is missing. Complete the code by filling the gaps (indicated by underscores) and send it to igor@malinovic.epfl.ch.

**Solution:**
The solution can be found in the GitHub repository of the course.

### Problem 3 ($\star$)

Suppose you are given an oracle algorithm, which for a given polyhedron

$$P = \{\tilde{x} \in \mathbb{R}^{\tilde{n}} \ : \ \tilde{A}\tilde{x} \leq \tilde{b}\}$$

gives you a feasible solution or asserts that there is none.
Consider the LP $\max\{c^T x \colon Ax \leq b,\, x \in \mathbb{R}^n\}$, and assume it is feasible and bounded. Show that one can obtain an optimal solution of the LP using a single call of the oracle algorithm on a suitable polyhedron. *Hint: Use duality theory!*

**Solution:**
The LP is feasible and bounded, thus an optimum solution must exist. Strong duality tells us that

the dual $\min\{b^T y : A^T y = c,\ y \geq 0\}$ is feasible and bounded. For optimal solutions $x^*$ of the primal and $y^*$ of the dual we have $b^T y^* = c^T x^*$.
Thus every point $(x^*, y^*)$ of the polyhedron

$$\begin{array}{rcl} c^T x & = & b^T y \\ Ax & \leq & b \\ A^T y & = & c \\ y & \geq & 0 \end{array}$$

is optimal. Hence with one oracle call for the polyhedron above we get an optimal solution of the LP.

**Problem 4**

Consider the following linear program:

$$\begin{array}{rc} \max & x_1 + x_2 \\ \text{subject to} & 2x_1 + x_2 \leq 6 \\ & x_1 + 2x_2 \leq 8 \\ & 3x_1 + 4x_2 \leq 22 \\ & x_1 + 5x_2 \leq 23 \end{array}$$

Show that $(4/3, 10/3)$ is an optimal solution by using weak duality.

**Solution:**

The assignment $(4/3, 10/3)$ has the objective function value of $14/3$. In order to prove that it is optimal (via strong duality), we are going to form the dual LP, and find a feasible solution to the dual that achieves the same objective value. The dual is:

$$\begin{array}{rcc} \min & 6y_1 + 8y_2 + 22y_3 + 23y_4 & \\ \text{subject to} & 2y_1 + y_2 + 3y_3 + y_4 = 1 & (1) \\ & y_1 + 2y_2 + 4y_3 + 5y_4 = 1 & (2) \\ & y_1, y_2, y_3, y_4 \geq 0 & \end{array}$$

Thus, we are looking for a feasible dual solution such that $6y_1 + 8y_2 + 22y_3 + 23y_4 = 14/3$. By using Gaussian elimination on this constraint combined with (1) and (2) we get:

$$-4y_2 - 2y_3 - 7y_4 = -4/3,$$

$$-3y_2 - 5y_3 - 9y_4 = -1$$

and further

$$14/3y_3 + 5y_4 = 0.$$

Since $y_1, y_2, y_3, y_4 \geq 0$, we have that $y_3 = y_4 = 0$ and then $y_1 = y_2 = 1/3$. This is the desired feasible dual solution coinciding with the primal solution $(4/3, 10/3)$, proving the optimality of the latter.

**Problem 5**

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ a bounded, non-empty polyhedron. Formulate a linear program that computes the largest ball inside $P$.

**Solution:**
A ball of radius $r$ and center $x$ is contained in $P$ if and only if $x \in P$ and $x$ has distance at least $r$ from any hyperplane defining $P$. Hence we obtain the following linear program:

$$\begin{array}{rl} \max & r \\ \text{subject to} & \frac{b_i - a_i x}{||a_i||} \geq r \quad \forall i = 1, \dots, m \\ & Ax \leq b \end{array}$$

where $a_1, \dots, a_m$ are the rows of $A$ and $b = (b_1 \dots b_m)^\top$.