École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

March 30, 2018

Discrete Optimization (Spring 2018)

Assignment 6

Problem 5 can be **submitted** until April 13 12:00 noon into the box in front of MA C1 563. You are allowed to submit your solutions in groups of at most three students.

Problem 1

Let $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$ be a polyhedron. Show that the following are equivalent:

i) x^* is a vertex of P.

ii) x^* is a basic feasible solution.

iii) For every feasible $x_1, x_2 \neq x^* \in P$ one has $x^* \notin \operatorname{conv}\{x_1, x_2\}$.

Solution:
i) $\Rightarrow$ iii) follows by Problem 3 in Assignment 3. Now, we prove by contradiction that iii) $\Rightarrow$ ii). Assume that iii) holds and let A_{x^*} be the submatrix of A corresponding to the constraints which are active at $x*$. If $\operatorname{rank}(A_{x^*}) < n$ then there exists a vector d and $\epsilon \in \mathbb{R}_{>0}$ such that $ad = 0$ for any row a of A_{x^*} and $x^* \pm \epsilon d \in P$. Since $x^* = 1/2(x^* + \epsilon d) + 1/2(x^* - \epsilon d)$ we have a contradiction. Finally, we prove that ii) $\Rightarrow$ i). Since x^* is a basic feasible solution it is the unique solution of the system $A_B x = b_B$ where B is a basis. Let $c = 1^T A_B$, then x^* is the unique point x such that $cx = 1^T b_B$ and $Ax \leq b$.

Problem 2

Prove or give a counter-example for the following statements:

i) Let B be an optimal basis. If λ_B is strictly positive then the optimal solution is unique.

ii) If the optimal solution is unique then λ_B is strictly positive for the optimal basis B.

Solution:
We consider the problem $\max\{c^{\mathrm{T}} x \ : x \in \mathbb{R}^n, Ax \leq b\}$.

i) True. Let x denote the basic feasible solution to B and $\lambda^{\mathrm{T}} A_B = c^{\mathrm{T}}$ with $\lambda > 0$. Assume there is another optimal solution $x' \neq x$. This gives

$$0 = c^{\mathrm{T}}(x - x') = \lambda^{\mathrm{T}}\left(A_B x - A_B x'\right) = \lambda^{\mathrm{T}}\left(b_B - A_B x'\right).$$

Since x' is feasible we have $A_B x' \leq b_B$ and since $x \neq x'$ there exists an index i such that $A_i x' < b_i$. This means all components of $(b_B - A_B x')$ are non-negative and one is strictly positive. Thus, $\lambda^{\mathrm{T}}(b_B - A_B x') > 0$, a contradiction.

ii) False. Consider the following linear program

$$\begin{aligned} \max \quad & x_2 \\ \text{subject to} \quad & -x_1 \leq 0 \\ & x_1 + x_2 \leq 1 \\ & x_2 \leq 1 \end{aligned}$$

This polyhedron has only one vertex, $(0,1)$ which is also the unique optimal solution. All inequalities are tight at $(0,1)$. Choosing $B = \{1,3\}$ gives

$$\lambda \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = [0 \quad 1],$$

and therefore $\lambda^{\mathrm{T}} = [0\ 1]$.

Problem 3

Prove that the truthfulness of the statement in Problem 2.ii) changes if we assume that the considered polyhedron is non-degenerate.

Solution:

We prove that the only counter-examples for the statement in Problem 2.ii) are degenerate polyhedra. Assume $P = \{x : Ax \leq b\}$ is a non-degenerate polyhedron, x^* the unique optimal solution to $\max\{c^T x : x \in P\}$ and that x^* is described by the basis B (B consists of all active constraints at x^* and is unique due to non-degeneracy).
Assume for the sake of contradiction that $\lambda = \lambda_B$ has a zero component $\lambda_j = 0$. Since A_B is invertible, we can choose a direction $d = (-1)A_B^{-1} e_j$ where e_j is the jth unit vector. We first show that there is a $\delta > 0$ such that $x^* + \delta d \in P$. Recall that the only constraints that are active/tight at x^* are in B. Hence, we can always choose δ small enough such that all the constraints outside B are not violated in $x^* + \delta d$. Now, consider the constraints in B:

$$A_B(x^* + \delta d) = A_B x^* + \delta A_B d \leq b_B - \delta e_j \leq b_B.$$

Thus, $x^* + \delta d \in P$.
Last, we prove that $x^* + \delta d$ is also an optimal solution:

$$c^T(x^* + \delta d) = c^T x^* + c^T \delta d = c^T x^* + \delta \lambda^T A_B d = c^T x^* - \delta \lambda^T e_j = c^T x^*,$$

where the last inequality follows because we assumed $\lambda_j = 0$. Hence, we have found another optimal and feasible solution and this contradicts the uniqueness of the optimum.

Problem 4

Assume you are the production manager for a lecture of an online course. For the production, n tasks have to be executed. A task j requires a working time of p_j hours to be completed. You have m employees at your disposal that can each, due to his or her qualifications, work on a subset of the tasks. Denote by S_i the set of jobs that employee i can work on.
As the production manager you want to create a work allocation plan that ensures that all tasks are completed. However, this allocation should also be fair. Consider the maximum number of working hours of each employee. You would like to minimize this quantity.
Model this problem as a linear program.

Solution:

We introduce variables x_{ij} to denote the working hours of employee i on task j. With a first constraint we would like to enforce that all tasks are processed, i.e., for every task j the sum of hours that the employees are working on it is at least p_j. Formally, we have $\sum_{i: j \in S_i} x_{ij} \geq p_j$ for all $j = 1, \ldots, n$. Furthermore, we note that the working hours cannot be negative, so $x_{ij} \geq 0$ for $i = 1, \ldots, m$ and $j \in S_i$. The objective is to minimize the function $\max_{i \in \{1,\ldots,m\}} \sum_{j \in S_i} x_{ij}$. However, this is not a linear function. The idea is to introduce a new variable t that stands for the maximum in the objective function. To do so, we add the constraints $t \geq \sum_{j \in S_i} x_{ij}$ for each $i = 1, \ldots, m$.

Eventually, we obtain the linear program:

$$\begin{aligned}
\min\ & t \\
t &\geq \sum_{j \in S_i} x_{ij} && i = 1, \ldots, m \\
\sum_{i: j \in S_i} x_{ij} &\geq p_j && j = 1, \ldots, n \\
x_{ij} &\geq 0 && i = 1, \ldots, m, \quad j \in S_i
\end{aligned}$$

This linear program can easily be translated into the inequality standard form.

Problem 5 ($\star$)

Consider the following linear program:

$$\begin{aligned}
\max \quad & 6a + 9b + 2c \\
\text{subject to} \quad & a + 3b + c \leq -4 && (1) \\
& b + c \leq -1 && (2) \\
& 3a + 3b - c \leq 1 && (3) \\
& a \leq 0 && (4) \\
& b \leq 0 && (5) \\
& c \leq 0 && (6)
\end{aligned}$$

Solve the linear program with the Simplex method and initial vertex $(-1, -1, 0)^T$. For each iteration indicate all the parameters including the optimal value and the proof of optimality.

Solution:

We give the result of the Simplex calculations below:

iteration	basis	vertex	λ	direction	ϵ	index exchange
1	$\{1, 2, 6\}$	$(-1, -1, 0)^T$	$(6, -9, 5)^T$	$(3, -1, 0)^T$	$1/3$	$2 \Rightarrow 4$
2	$\{1, 4, 6\}$	$(0, -4/3, 0)^T$	$(3, 3, -1)^T$	$(0, 1/3, -1)^T$	$5/2$	$6 \Rightarrow 3$
3	$\{1, 3, 4\}$	$(0, -1/2, -5/2)^T$	$(5/2, 1/2, 2)^T$			

The last vertex is optimal with an objective function value of $-19/2$.