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# Combinatorial Optimization

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Assignment Sheet 5

**Exercise 1**
Let $G = (V, E)$ be an undirected graph with edge weights $c : E \to \mathbb{R}_+$. Consider the recursive algorithm to compute a Gomory-Hu tree. Modify it such that instead of taking a minimum cut $\delta(W)$ among all cuts separating at least two vertices of $R$, it instead chooses two arbitrary vertices $s, t \in R$ and takes a minimum $s$-$t$ cut $\delta(W)$. The rest of the algorithm proceeds unchanged.

1. Prove that $\delta(W)$ is a minimum $r'$-$r''$ cut.

   *Hint:* You can assume by contradiction that there exists a lighter cut and then consider the $s - r'$ path in $T'$ (or the $t - r''$ path in $T''$).

2. Prove that the modified algorithm computes a Gomory-Hu tree.

**Exercise 2 (⋆)**
Let $G = (V, E)$ be an undirected graph with requirements $r : E \to \mathbb{R}_+$. The *minimum-requirement spanning tree problem* is the problem of finding a spanning tree $T$ on the vertices $V$ such that $\sum_{e \in E} r(e) \cdot d_T(e)$ is minimized, where $d_T(e)$ is the distance (number of edges) between the endpoints of $e$ in the tree $T$.

1. For any edge $f \in T$, let $R_T(f)$ denote the weight of the cut in $G$ induced by the two components of $T \setminus \{f\}$. Show that $\sum_{e \in E} r(e) \cdot d_T(e) = \sum_{f \in T} R_T(f)$.

2. Let $T$ and $T'$ be spanning trees on $V$. Show that there exists a bijection $\phi : T \to T'$ between the edges such that for all $e \in T$, $\phi(e)$ is an edge on the unique path in $T'$ connecting the endpoints of $e$.

3. Show that a Gomory-Hu tree is an optimal solution to the minimum-requirement spanning tree problem.

*Remark:* This problem is motivated by a network design question. The requirements indicate desired bandwidth between sites, and the goal is to satisfy them with a tree-shaped network such that the total capacity of the network is as small as possible to save costs.

**Exercise 3**
Let $E$ be a finite set and let $\mathscr{I}$ be a non-empty collection of subsets of $E$ such that $I \in \mathscr{I}$ and $J \subseteq I$ implies $J \in \mathscr{I}$. Prove that the following conditions are equivalent:

1. if $I, J \in \mathscr{I}$ and $|J| > |I|$, then $I \cup \{e\} \in \mathscr{I}$ for some $e \in J \setminus I$;
2. if $I, J \in \mathscr{I}$ and $|J| = |I| + 1$, then $I \cup \{e\} \in \mathscr{I}$ for some $e \in J \setminus I$;
3. if $I, J \in \mathscr{I}$ and $|I \setminus J| = 1$, $|J \setminus I| = 2$, then $I \cup \{e\} \in \mathscr{I}$ for some $e \in J \setminus I$.

**Exercise 4 (⋆)**
Let $E$ be a finite set that is partitioned into sets $E = E_1 \cup \ldots \cup E_r$ and define a system

$$\mathscr{I} := \{S \subset E \mid |S \cap E_j| \leq 1 \text{ for all } j = 1 \ldots r\}.$$

of independent sets. Show that $(E, \mathscr{I})$ is a matroid. What is the rank of this matroid? Give a simple description of the bases of the matroid.
*Remark:* This type of matroid is called a *partition matroid*.

**Exercise 5**
Let $D = (V, A)$ be a directed graph. A Hamiltonian $s$-$t$-path $P \subset A$ is a simple path from $s$ to $t$ that contains every vertex of the graph. A Hamiltonian cycle $C \subset A$ is a simple cycle that contains every vertex of the graph.

1. Let $s, t \in V$ be distinct vertices. Find an intersection $M$ of three matroids so that Hamiltonian $s$-$t$-paths in $D$ are exactly the elements of $M$ with $|V| - 1$ elements.
2. Find an intersection $M$ of three matroids so that Hamiltonian cycles in $D$ are exactly the elements of $M$ with $|V|$ elements.
3. Conclude that, given an oracle that can optimize over arbitrary matroid intersections, we can find a Hamiltonian cycle in $D$. If the oracle supports weights on the edges, this can be used to solve TSP.