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Combinatorial Optimization

Fall 2010

Assignment Sheet 4

Exercise 1

Show that the maximum number of minimum cuts that an undirected graph on n vertices can have is $\binom{n}{2}$. That is, give an example of an infinite family of graphs with this many minimum cuts, and prove that the number of minimum cuts cannot be larger.

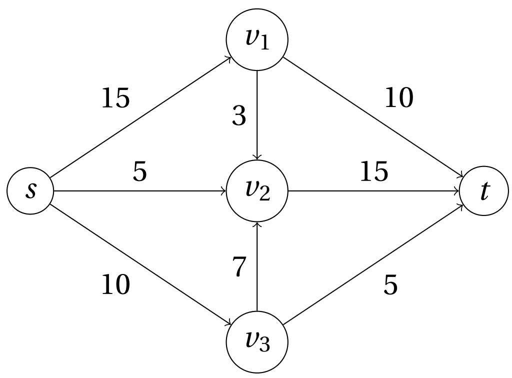
Exercise 2 ($\star$)

Let $G = (V, E)$ be an undirected graph with edge weights $w : E \to \mathbb{R}_+$. Let α be a positive integer. We say that a cut $A \subset E$ is α-approximate iff $w(A) \le \alpha\lambda(G)$, where $\lambda(G)$ is the weight of a minimum cut.

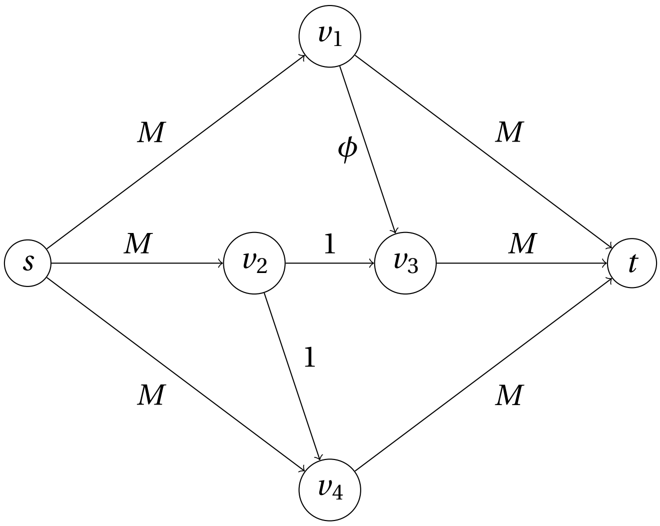
1. Show that after $n - 2\alpha$ iterations of the random contraction (Karger's) algorithm, any α-approximate cut A is still present with probability at least $\binom{n}{2\alpha}^{-1}$.
2. Consider the following algorithm for computing α-approximate cuts. Do $n - 2\alpha$ iterations of the random contraction algorithm, and then output any cut of the remaining graph with equal probability. Give as good as possible a lower bound on the probability that this algorithm output a given α-approximate cut.

Exercise 3

Run the Edmonds-Karp algorithm to find a maximum s-t-flow in the following network. Indicate the final residual network and a minimum s-t-cut.



Exercise 4
Let M be some number and let ϕ be the golden ratio, i.e. the positive number satisfying $\phi^2 = 1 - \phi$. Show that, given an appropriate choice of augmenting path steps and M big enough, the Ford-Fulkerson algorithm will not terminate in the following network. In fact, show that the value of the computed flow does not converge against a maximum flow.



Hint: Start with the path $s - v_2 - v_3 - t$. Then keep track of the residual capacities on the three crucial edges, and find a repeating sequence of augmenting paths that will decrease the residual capacities geometrically. There exists such a sequence of length 4.

Exercise 5 (⋆)
Let $G = (V, E)$ be an undirected graph with an even number of vertices and edge weights $w : E \to \mathbb{R}_+$. Recall that a Gomory-Hu tree for G is a tree T on the vertices V such that for every edge $e = uv \in T$, the cut which is induced by the two connected components of $T \setminus \{e\}$ is a minimum u-v-cut in G. We will show that the minimum odd cut problem of G can be solved using a Gomory-Hu tree.

1. Let $U \subset V$ be a minimum odd cut. Consider the induced cut $\delta_T(U)$ of the Gomory-Hu tree. Show that the forest $T \setminus \delta_T(U)$ has an odd connected component.

2. Show that there exists an edge $e^\star \in \delta_T(U)$ such that $e^\star$ splits T into two odd connected components.

3. Show that the cut induced by $e^\star$ is a minimum odd cut of G.

4. Describe an algorithm that, given an undirected graph G with edge weights w and a Gomory-Hu tree T for G, finds a minimum odd cut in polynomial time. Prove the correctness and analyze the running time of your algorithm.