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Due Date: April 15, 2010
Discussions: March 25, April 01

Discrete Optimization

Spring 2010

Assignment Sheet 3

You can hand in written solutions for up to two of the exercises marked with (*) or (Δ) to obtain bonus points. The duedate for this is April 15, 2010, before the exercise session starts. Math students are restricted to exercises marked with (*). Non-math students can choose between (*) and (Δ) exercises.

Exercise 1 (*)
Prove the following statement:

Let $A \in \mathbb{R}^{m\times n}$ be a matrix and $b \in \mathbb{R}^m$ be a vector. The system $Ax = b$, $x \geq 0$ has a solution if and only if for all $\lambda \in \mathbb{R}^m$ with $\lambda^T A \geq 0$ one has $\lambda^T b \geq 0$.

Hint: Use duality theory!

Exercise 2

1. Consider the linear program
$$\max\{c^T x \,:\, Ax \leq b\}$$
with $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^n$. The dual of the linear program is
$$\min\{b^T y \,:\, A^T y = c,\ y \geq 0\}.$$
Let x and y be feasible solutions to the primal and dual LP, respectively. Prove the following statement:
The vectors x and y are optimal solutions to their respective LPs if and only if for each $i = 1, \ldots, m$ we have $a_i^T x = b_i$ or $y_i = 0$.
Hint: Consider the inner product $(b - Ax)^T y$ and show that it is 0 if and only if x and y are optimal. Why is this sufficient to show?

2. Consider the following linear program:
$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 6 \\ & x_1 + 2x_2 \leq 8 \\ & 3x_1 + 4x_2 \leq 22 \\ & x_1 + 5x_2 \leq 23 \end{aligned}$$
Show that $(4/3, 10/3)$ is an optimal solution by computing an optimal solution of the dual using the first part of this exercise.

Exercise 3
Consider the linear program
$$\max\{c^T x \,:\, Ax \leq b\}$$
with $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^n$. The dual is
$$\min\{b^T y \,:\, A^T y = c,\ y \geq 0\}.$$
The following table describes nine situations for the status of primal and dual LPs. Decide which one of these situations are possible/impossible. If they are possible, give an example. Otherwise give a short argument why they are impossible.

		Dual		
		Finite optimum	Unbounded	Infeasible
Primal	Finite optimum			
	Unbounded			
	Infeasible			

Exercise 4
Suppose you are given an oracle algorithm, which for a given polyhedron
$$P = \{\tilde{x} \in \mathbb{R}^{\tilde{n}} \,:\, \tilde{A}\tilde{x} \leq \tilde{b}\}$$
gives you a feasible solution or asserts that there is none. Show that using a *single* call of this oracle one can obtain an *optimum* solution for the LP
$$\max\{c^T x \,:\, x \in \mathbb{R}^n;\ Ax \leq b\},$$
assuming that the LP is feasible and bounded.
Hint: Use duality theory!

Exercise 5 (Δ)
A *directed graph* $D = (V, A)$ is a tuple consisting of a set of *vertices* V and a set of *arcs* $A \subseteq V \times V$. Given an arc $a = (u, v) \in A$, the vertex u is called the *tail* of a and v is called the *head* of a. For some vertex $v \in V$, the set of arcs whose tail is v is called
$$\delta^{out}(v) := \{a \in A \,:\, a = (v, u) \text{ for some } u \in V\}.$$
Analogous, the set of arcs whose head is v is called
$$\delta^{in}(v) := \{a \in A \,:\, a = (u, v) \text{ for some } u \in V\}.$$
Let $s, t \in V$ and let $c : A \to \mathbb{N}_0$ be a capacity function. A function $f : A \to \mathbb{N}_0$ is an $s-t$-*flow*, if for every vertex $v \in V \setminus \{s, t\}$ we have *flow conservation*, i.e.
$$\sum_{a\in\delta^{in}(v)} f(a) - \sum_{a\in\delta^{out}(v)} f(a) = 0.$$

Moreover, $0 \le f(a) \le c(a)$ must hold for each $a \in A$.

The *value* of a flow f is defined as $\sum_{a \in \delta^{in}(t)} f(a) - \sum_{a \in \delta^{out}(t)} f(a)$. Formulate the problem of finding an $s-t$-flow of maximum value as a linear program.

Hint: Use the so called node-arc incidence matrix. It has a row for each vertex and a column for each arc. Each column consists of zeros, except for the entries corresponding to the head and the tail of the arc: The head gets a (1) *entry and the tail gets a* (-1) *entry.*

Exercise 6 (*)

In the lecture you have seen the simplex algorithm for linear programs of the form

$$\max\{c^T x \,:\, Ax \le b\}.$$

We will now derive a simplex algorithm for linear programs of the form

$$\min\{c^T x \,:\, Ax = b,\ x \ge 0\} \tag{1}$$

with $c \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Throughout the exercise we assume that (1) is feasible and bounded, and that A has full row rank.

For $i \in \{1, \dots, n\}$ we define A_i as the i-th column of A. Moreover, for some subset $B \subseteq \{1, \dots, n\}$, A_B is the matrix A restricted to the columns corresponding to elements of B.

A subset $B \subseteq \{1, \dots, n\}$ with $|B| = n$ such that A_B has full rank is called a *basis*. The vector $x \in \mathbb{R}^n$ defined as $x_i := 0$ for all $i \notin B$ and $x_B := A_B^{-1} b$ is called the *basic solution* associated to B. Note that x is a feasible solution to (1) if and only if $x \ge 0$.

Given a basis B and let $j \in \{1, \dots, n\}$, $j \notin B$. The vector $d \in \mathbb{R}^n$ defined as $d_j = 1$, $d_i = 0$ for all $i \notin B$ and $d_B := -A_B^{-1} A_j$ is called the *j-th basic direction*.

Assume that the solution x associated to B is feasible. Moreover assume that $x_B > 0$.

1. Show that there is a $\theta > 0$ such that $x + \theta d$ is a feasible solution. Give a formula to compute the largest θ such that $x + \theta d$ is feasible.

2. Let θ^* be maximal. Show that there is a basis B' such that $x + \theta^* d$ is the basic solution associated to B'.

3. Let $x' = x + \theta d$. Show that the objective value of x' changes by $\theta \left(c_j - c_B^T A_B^{-1} A_j\right)$.

4. Consider a basis B with basic feasible solution x. Show that if $c - c_B^T A_B^{-1} A \ge 0$, then x is an optimal solution to (1).

This suggests the following algorithm: Start with some basis B whose associated basic solution is feasible. Compute $\bar{c} := c - c_B^T A_B^{-1} A$. If $\bar{c} \ge 0$, we have an optimal solution (see 4). Otherwise, let j be such that $\bar{c}_j < 0$. Part 2 and 3 show that if we change the basis, we find a feasible solution with an improved objective value. We repeat these steps until the vector $\bar{c}$ is nonnegative.

This is the way the simplex algorithm usually is introduced in the literature. This algorithm is exactly the same as the one you learned in the lecture. To get an intuition why this is true, show the following:

5. Given a basis B, show that its associated basic solution is feasible if and only if B is a *roof* of the LP dual to (1).

6. Consider a basis B and its associated feasible basic solution x. As seen before, B is also a roof in the dual LP. Let y be the vertex of that roof.

   Show that for any $j \in \{1, \ldots, n\}$ we have $\bar{c}_j < 0$ if and only if $A_j^T y > c_j$.