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Due Date: March 25, 2010
Discussions: March 11, March 18

Discrete Optimization

Spring 2010

Assignment Sheet 2

You can hand in written solutions for up to two of the exercises marked with (*) or (Δ) to obtain bonus points. The duedate for this is March 25, 2010, before the exercise session starts. Math students are restricted to exercises marked with (*). Non-math students can choose between (*) and (Δ) exercises.

Exercise 1
A factory produces two different products. To create one unit of product 1, it needs one unit of raw material A and one unit of raw material B. To create one unit of product 2, it needs one units of raw material B and two units of raw material C. Raw material B needs preprocessing before it can be used, which takes one minute per unit. At most 20 hours of time is available per day for the preprocessing. Raw materials of capacity at most 1200 can be delivered to the factory per day. One unit of raw material A, B and C has size 4, 3 and 2 respectively.

At most 130 units of the first and 100 units of the second product can be sold per day. The first product sells for 6 CHF per unit and the second one for 9 CHF per unit.

Formulate the problem of maximizing turnover as a linear program in two variables.

Exercise 2
Consider a school district with I neighborhoods, J schools and G grades at each school. Each school j has a capacity of C_{jg} for grade g. In each neighborhood i, the student population of grade g is S_{ig}. Finally the distance of school j from neighborhood i is d_{ij}.

We want to solve the following problem: Assign all students to schools, in such a way that this assignment does not exceed the capacity of the schools. Moreover, the total distance travelled by all students should be minimal.

1. Try to formulate this as a linear program. What difficulties arise?
2. Download an instance for this problem here:

   `http://disopt.epfl.ch/webdav/site/disopt/shared/Opt2010/schools_inst.zmpl`

3. Model the problem using ZIMPL and solve the instance using an LP solver of your choice. What property has the optimal solution for the linear program? What does this tell you about an optimal solution for the problem?

Exercise 3
A cone $C \subseteq \mathbb{R}^n$ is *pointed* if it does not contain a line: There are no vectors $x \in C$, $v \in \mathbb{R}^n$ such that $x + \lambda v \in C$ for all $\lambda \in \mathbb{R}$.

Prove the following variant of Carathéodory's theorem: Given some set $X \subseteq \mathbb{R}^n$, $|X| > n$ such that $\text{cone}(X)$ is pointed. For any $x \in \text{cone}(X)$, there exist at least two different subsets $X_1, X_2 \subseteq X$ with $|X_1| = |X_2| = n$ such that $x \in \text{cone}(X_1) \cap \text{cone}(X_2)$.

Exercise 4
A polyhedron $P = \{x \in \mathbb{R}^n \colon Ax \le b\}$ *contains a line*, if there exists a nonzero $v \in \mathbb{R}^n$ and an $x^* \in R^n$ such that for all $\lambda \in \mathbb{R}$, the point $x^* + \lambda \cdot v \in P$. Show that a nonempty polyhedron P contains a line if and only if A does not have full column-rank.

Exercise 5 (*)
Consider a linear program

$$\max\{c^T x \,:\, Ax \le b\}.$$

Let B be a roof for the linear program.

1. Let $j \in B$, and consider the following system of equations:

$$\begin{aligned} a_j^T v &= -1 \\ a_i^T v &= 0 \quad \forall i \in B,\ i \neq j. \end{aligned} \tag{1}$$

   Prove the following statement: If x is a feasible solution for the *roof linear program* and v is a solution to system (1), then for all $\lambda > 0$, the vector $x + \lambda v$ is a feasible solution for the roof linear program as well.

2. Prove the following statement: The *vertex* of the root B is the *unique* optimal solution to the roof linear program if and only if c is a *strictly positive* conic combination of the vectors a_i of the roof.

Exercise 6 (Δ)
Solve the following linear program using the simplex method:

$$\begin{aligned} \max \quad & 6x + 9y + 2z \\ \text{subject to} \quad & x + 3y + z \le -4 \\ & y + z \le -1 \\ & 3x + 3y - z \le 1 \\ & x \le 0 \\ & y \le 0 \\ & z \le 0 \end{aligned}$$

Start with the roof given by the three last rows. For each iteration, give the vertex of the roof, the row which leaves the roof and the row which enters the roof (together with an argument why you can choose the rows as you did).

Exercise 7 (*)
Consider the problem

$$
\begin{aligned}
\max \quad & z \\
\text{subject to} \quad & x + 2y \le -3 && (2)\\
& -2x - 3y \le 5 && (3)\\
& -2x - y + 2z \le -1 && (4)\\
& 3x + y \le 2 && (5)\\
& x \le 0 && (6)\\
& y \le 0 && (7)\\
& z \le 0. && (8)
\end{aligned}
$$

Assume that we perform the simplex method, and at some point have the roof given by the rows (2), (7) and (8). The illustration below shows the situation in the 2-dimensional subspace given by the hyperplane $z = 0$.

Show that the simplex algorithm might not terminate, by giving a cycling sequence of roofs might be selected by the simplex method. Explain why your sequence is valid (it is sufficient to give drawings here, you do not need to compute the roof vertices explicitly).

Hint: Never let (8) leave the roof. Then it is sufficient to consider the subspace as in the illustration.