École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

February 24, 2017

Discrete Optimization (Spring 2017)

Assignment 2

Problem 5 can be **submitted** until March 10 12:00 noon into the right box in front of MA C1 563.
You are allowed to submit your solutions in groups of at most two students.

Problem 1

Show that the recursion $T(n) = 8 \cdot T(n/2) + \theta(n^2)$, with the initial condition $T(1) = \theta(1)$, has the solution $T(n) = \theta(n^3)$.

Problem 2

Describe an algorithm that multiplies two n-bit integers in time $O(n^2)$. You may use the algorithm to add two n-bit integers from Assignment 1, Problem 7.

Problem 3

Suppose $n = 2^\ell$ and a, b are two n-bit integers. Consider the numbers a_h and a_l which are represented by the first $n/2$ bits and the last $n/2$ bits of a respectively. Likewise the numbers b_h and b_l are the numbers represented by the first half and the second half of the bit-representation of b.

i) Show $a = a_h 2^{n/2} + a_l$ and $b = b_h 2^{n/2} + b_l$

ii) Show $ab = a_h b_h 2^n + (a_h b_l + a_l b_h) 2^{n/2} + a_l b_l$

iii) Conclude very carefully that two n-bit numbers can be multiplied by resorting to three multiplications of $n/2$-bit numbers and $O(n)$ basic operations.

iv) Conclude that two n-bit numbers can be computed in time $O(n^{\log_2(3)})$ elementary bit operations.

Problem 4

Given a random $n \times n$ matrix M where each entry is i.i.d. variable taking the value 1 or -1 with the equal probability $1/2$, show the following:

$$Pr\left[\det(M) = 0\right] \geq (1 - o(1)) n^2 \frac{1}{2^{n+1}} \tag{1}$$

Problem 5 ($\star$)

Let M_{2^k} be a matrix of order $n := 2^k$, where $k \in \mathbb{N}_{>0}$ such that it is recursively defined as follows:

$$M_{2^k} = \begin{pmatrix} M_{2^{k-1}} & M_{2^{k-1}} \\ M_{2^{k-1}} & -M_{2^{k-1}} \end{pmatrix} \tag{2}$$

and $M_1 = [1]$. Prove that $|\det(M_n)| = n^{n/2}$, i.e. that the Hadamard bound is tight.