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Due Date: March 11, 2010
Discussions: February 25, March 04

Discrete Optimization

Spring 2010

Assignment Sheet 1

You can hand in written solutions for up to two of the exercises marked with (*) or (Δ) to obtain bonus points. The duedate for this is March 11, 2010, before the exercise session starts. Math students are restricted to exercises marked with (*). Non-math students can choose between (*) and (Δ) exercises.

Exercise 1

A company produces and sells two different products. Our goal is to determine the number of units of each product they should produce during one month, assuming that there is an unlimited demand for the products, but there are some constraints on production capacity and budget.

There are 20000 hours of machine time in the month. Producing one unit takes 3 hours of machine time for the first product and 4 hours for the second product. Material and other costs for producing one unit of the first product amount to 3CHF, while producing one unit of the second product costs 2CHF. The products are sold for 6CHF and 5CHF per unit, respectively. The available budget for production is 4000CHF initially. 25% of the income from selling the first product can be used immediately as additional budget for production, and so can 28% of the income from selling the second product.

1. Formulate a linear program to maximize the profit subject to the described constraints.
2. Solve the linear program graphically by drawing its set of feasible solutions and determining an optimal solution from the drawing.
3. Suppose the company could modernize their production line to get an additional 2000 machine hours for the cost of 400CHF. Would this investment pay off?

Exercise 2

Consider the problem

$$\begin{aligned} \min \quad & 2x + 3|y-10| \\ \text{subject to} \quad & |x+2| + y \leq 5, \end{aligned} \tag{1}$$

and reformulate it as a linear programming problem.

Exercise 3
Prove the following statement or give a counterexample: The set of optimal solutions of a linear program is always finite.

Exercise 4
Let (2) be a linear program in inequality standard form, i.e.

$$\max\{c^T x \mid Ax \leq b, x \in \mathbb{R}^n\} \tag{2}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.

Prove that there is an equivalent linear program (3) of the form

$$\min\{\tilde{c}^T x \mid \tilde{A}x = \tilde{b}, x \geq 0, x \in \mathbb{R}^{\tilde{n}}\} \tag{3}$$

where $\tilde{A} \in \mathbb{R}^{\tilde{m}\times\tilde{n}}$, $\tilde{b} \in \mathbb{R}^{\tilde{m}}$, and $\tilde{c} \in \mathbb{R}^{\tilde{n}}$ are such that every feasible point of (2) corresponds to a feasible point of (3) with the same objective function value and vice versa.

Linear programs of the form in (3) are said to be in *equality standard form*.

Exercise 5
Recall the image decomposition problem for OLEDs from the lecture and its formulation as a linear program :

$$\begin{aligned}
\min \quad & \sum_{i=1}^{n} u_i^{(1)} + \sum_{i=1}^{n-1} u_i^{(2)} && \qquad (4)\\
\text{s.t.} \quad & f_{ij}^{(1)} + f_{i-1,j}^{(2)} + f_{ij}^{(2)} = r_{ij} && \text{for all } i,j\\
& f_{ij}^{(\alpha)} \leq u_i^{(\alpha)} && \text{for all } i,j,\alpha\\
& f_{ij}^{(\alpha)} \geq 0 && \text{for all } i,j,\alpha
\end{aligned}$$

Apply This technique to find an optimal decomposition of the EPFL logo[1] using Zimpl and LP solver libraries:

1. Familiarize yourself with the Zimpl modelling language[2].
2. Model the linear program (4) to decompose the EPFL logo with Zimpl. You can find an incomplete model containing the encoding of the grayscale values of the logo here[3].
3. Solve the linear program using an LP-solver like QSopt[4], lp_solve[5] or SoPlex[6].

[1] http://www.epfl.ch/images/EPFL-logo.jpg
[2] http://www.zib.de/koch/zimpl/download/zimpl.pdf
[3] http://disopt.epfl.ch/webdav/site/disopt/users/190205/public/logo_dec.zmpl
[4] http://www2.isye.gatech.edu/ wcook/qsopt/
[5] http://lpsolve.sourceforge.net/5.5/
[6] http://soplex.zib.de/

Exercise 6 (*)

1. Let $\{C_i\}_{i\in I}$ be a family of convex subsets of $\mathbb{R}^n$. Prove that the intersection $\bigcap_{i\in I} C_i$ is convex.

2. Let $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Prove that the *closed halfspace* $H := \{x \in \mathbb{R}^n \mid a^T x \leq b\}$ is closed and convex.

3. Prove that the set of feasible points of a linear program is convex and closed.

4. Find a convex and closed set $C \subset \mathbb{R}^2$ and a vector $c \in \mathbb{R}^2$ such that $\sup\{c^T x \mid x \in C\}$ is finite but $\max\{c^T x \mid x \in C\}$ does not exist.

Exercise 7 (Δ)
Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map and let $K \subseteq \mathbb{R}^n$ be a set.

1. Show that $f(K) := \{f(x) : x \in K\}$ is convex if K is convex. Is the reverse also true?

2. Prove that $\text{conv}(f(K)) = f(\text{conv}(K))$.

Exercise 8 (*)
Consider the vectors

$$x_1 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, x_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, x_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, x_5 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The vector

$$v = x_1 + 3x_2 + 2x_3 + x_4 + 3x_5 = \begin{pmatrix} 15 \\ 14 \\ 25 \end{pmatrix}$$

is a conic combination of the x_i.
Write v as a conic combination using only three vectors of the x_i.

Hint: Recall the proof of Carathéodory's theorem