Computer Algebra

Spring 2013

Assignment Sheet 1

Exercises marked with a $\star$ can be handed in for bonus points. Due date is March 5.

Exercise 1

Sort the following functions according to their asymptotic growth. Indicate which pairs of functions satisfy $f = O(g)$, $f = \Omega(g)$, and $f = \Theta(g)$, motivating your answer.

$2^{3+\log n}$, $\sqrt{n}$, $\log n^n$, 4^n, 13, $\log n^{1337}$, $2^{\log^2 n}$, $\log n$, $e^{\log n}$, $3n$, $n^6 - 5n^2$, $-n^6 + 5n^2$, $2^{4\log n}$, 2^n, $\log^2 n$

Note: $\log n$ without an indicated base is always base 2.

Exercise 2

Let $f, g : \mathbb{N} \to \mathbb{R}_+$. Show that $f = O(g)$ if and only if $\limsup_{n \to \infty} \frac{f(n)}{g(n)} < \infty$.

Exercise 3

Let $f, g : \mathbb{N} \to \mathbb{R}_+$. We say $f \sim g$ (f is asymptotically equal to g) when $f(x)/g(x) \to 1$ as $x \to \infty$.

a) [$\star$] Show that $f \sim g$ implies $f = \Theta(g)$. Is the converse also true?

b) Show that $f \sim g$ implies $f = (1 + o(1))g$. Is the converse also true?

c) [$\star$] Let $F(n) = \sum_{i=1}^{n} f(i)$ and $G(n) = \sum_{i=1}^{n} g(i)$. Show that $f \sim g$ and $G(n) \to +\infty$ when $n \to \infty$ implies $F \sim G$.

Exercise 4 ($\star$)

Let $f(n) = n \log n$ and $g(n) = \log(n!)$. Show which among the following relations is true: $f = O(g)$; $f = \Omega(g)$; $f = \Theta(g)$.

Exercise 5

Let A_1 and A_2 be algorithms for the same problem which run for $T_1(n) = 5n^2$ and $T_2(n) = 1000 n \log n$ machine operations on an input of size n, respectively. Let M_1 be a machine that can execute 10^{10} machine operations per second, and M_2 a machine that can execute 10^6 machine operations per second. For which values of n is A_1 on M_1 faster than A_2 on M_2?