Prof. Friedrich Eisenbrand
Dr. Linda Farczadi

Discussions from: October 28, 2015

Combinatorial Optimization

Fall 2015

Assignment Sheet 7

Exercises marked with $\star$ can be handed in for bonus points. Due date is Friday November 6.

Exercise 1

Recall that in class we defined a chain $\mathscr{C}$ to be a family of sets such that for all $S, T \in \mathscr{C}$ we have either $S \subseteq T$ or $T \subseteq S$. Suppose $\mathscr{C}_1$ and $\mathscr{C}_2$ are two chains. Let A be the matrix with rows χ^S for all $S \in \mathscr{C}_1 \cup \mathscr{C}_2$. Prove that A is totally unimodular. That is, show that for all square submatrices B of A we have $\det(B) \in \{0, \pm 1\}$.

Exercise 2

Recall that for a chain $\mathscr{C}$ and a set S we defined

$$\text{viol}(S) = \{C \in \mathscr{C} : C \nsubseteq S \text{ and } S \nsubseteq C\}$$

Show that for any $C' \in \text{viol}(S)$ we have

$$|\text{viol}(S)| > |\text{viol}(S \cap C')| \quad \text{and} \quad |\text{viol}(S)| > |\text{viol}(S \cup C')|.$$

Exercise 3 ($\star$)

Two vertices x, x' of a polyhedron P are said to be *adjacent* if they are contained in a face F of P of dimension one.

Let $M = (E, \mathscr{I})$ be a matroid and P_M the corresponding matroid polytope:

$$P_M := \text{conv}\{\chi^S : S \in \mathscr{I}\}$$

Given $I_1, I_2 \in \mathscr{I}$ with $I_1 \neq I_2$, show that χ^{I_1} and χ^{I_2} are adjacent vertices of P_M if and only if one of the following conditions hold:

(i) $I_1 \subseteq I_2$ and $|I_1| + 1 = |I_2|$

(ii) $I_2 \subseteq I_1$ and $|I_2| + 1 = |I_1|$

(iii) $|I_1 \setminus I_2| = |I_2 \setminus I_1| = 1$ and $I_1 \cup I_2 \notin \mathscr{I}$

Exercise 4

Given a directed graph $D = (V, A)$ and a special root vertex $r \in V$ an r*-arborescence* is a subset of arcs $B \subseteq A$ such that for each vertex $v \in V \setminus \{r\}$ there is a unique directed path from r to v in (V, B). Given a cost function $c : A \to \mathbb{R}$ we are interested in the problem of finding an r-arborescence of minimum cost. Show how this problem can be stated as a matroid intersection problem.