Summary of last two lectures on linear Programming

Primary objectives:

- Linear programming: Example, Definition
- The standard form
- Basic feasible solutions
- simplex algorithm
- Along the way: Develop your own toy-solver in C++
- Use of modeling languages and efficient LP-solvers
- Constructing dedicated bond-portfolio

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Software

ZIMPLE:

- A modelling language for linear/integer programs
- Enables compact formulations of optimization problems
- Command `zimpl` converts `zpl` files to `lp` or `mps` files which can be solved with most solvers

Soplex:

- LP solver for Windows/Linux/MacOS
- Can solve problems, described as `lp` or `mps` file

The software is installed in the terminal room. You can also download it here. The software is free for academic purposes (means free for us!).

Modelling with ZIMPL

Recall Cashflow Management Problem

Our company has net cash flow requirements (in 1000 CHF)

Month	Jan	Feb	Mar	April	May	June
Net cash flow	-150	-100	200	-200	50	300

For example in January we have to pay CHF 150k, while in March we will get CHF 300k.

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For example in January we have to pay CHF 150k, while in March we will get CHF 300k. Initially we have 0 CHF, but the following possibilities to borrow/invest money

1. We have a credit line of CHF 100k per month. For a credit we have to pay 1% interest per month.
2. In the first 3 months we may issue a 90-days commercial paper (to borrow money) at a total interest of 2% (for the whole 3 months).
3. Excess money can be invested at an interest rate of 0.3% per month.

Task: We want to maximize our wealth in June, while we have to fulfill all payments. How shall we invest/borrow money?

Modelling with ZIMPL

Linear Program::

Define $b = (-150, -100, 200, -200, 50, 300)$ as the vector of net cash flow. Then the following LP determines the optimal investment strategy:

$$
\begin{aligned}
\max z_6 - x_6 & & & \\
x_1 + y_1 - z_1 &= -b_1 & & \\
x_i + y_i - 1.01x_{i-1} + 1.003z_{i-1} - z_i &= -b_i & & \forall i = 2,3 \\
x_i - 1.02y_{i-3} - 1.01x_{i-1} + 1.003z_{i-1} - z_i &= -b_i & & \forall i = 4,\dots,6 \\
0 \leqslant x_i &\leqslant 100 & & \forall i = 1,\dots,6 \\
z_i &\geqslant 0 & & \forall i = 1,\dots,6 \\
y_i &\geqslant 0 & & \forall i = 1,2,3
\end{aligned}
$$

Modelling with ZIMPL

File: `cashflow.zpl`

```
set Months := {1,2,3,4,5,6};
param b[Months] :=< 1 > -150, < 2 > -100, < 3 > 200,
                  < 4 > -200, < 5 > 50, < 6 > 300;

var x[Months] real >= 0 <= 100;
var y[{1,2,3}] real >= 0;
var z[Months] real >= 0;

maximize wealth : z[6] - x[6];
subto Jan : x[1] + y[1] - z[1] == -b[1];
subto FebToMar : forall < i >  in {2..3} do
 x[i] + y[i] - 1.01 * x[i - 1] + 1.003 * z[i - 1] - z[i] == -b[i];
subto AprToJun : forall < i > in{4..6} do
 x[i] - 1.02 * y[i - 3] - 1.01 * x[i - 1] + 1.003 * z[i - 1] - z[i] == -b[i];
```

Running `zimple cashflow.zpl` creates `cashflow.lp`

The LP File

File cashflow.lp

```
\This file was automatically generated by Zimpl
...
Maximize
 wealth: -x#6 + z#6
Subject to
Jan_1:   -z#1 + y#1 + x#1 = 150
FebToMar_1: -z#2 + 1.003z#1 - 1.01x#1 + y#2 + x#2 = 100
FebToMar_2: -z#3 + 1.003z#2 - 1.01x#2 + y#3 + x#3 = -200
AprToJun_1: -z#4 + 1.003z#3 - 1.01x#3 - 1.02y#1 + x#4 = 200
AprToJun_2: -z#5 + 1.003z#4 - 1.01x#4 - 1.02y#2 + x#5 = -50
AprToJun_3: -z#6 + 1.003z#5 - 1.01x#5 - 1.02y#3 + x#6 = -300
Bounds
0 <= x#1 <= 100
...
0 <= z#6 <= +inf
End
```

Solving the LP File with Soplex

Results of `soplex -x cashflow.lp`

```
 SoPlex statistics:
  Factorizations : 3
      Time spent : 0.00
  Solves         : 32
      Time spent : 0.00
  solution time  : 0.00
  iterations     : 12

Solution value is: 9.2496949e+01
z#6     1          9.2496949e+01
y#1     3          1.5000000e+02
y#2     6          1.0000000e+02
z#3     8          3.5194417e+02
y#3     9          1.5194417e+02
x#5     14         5.2000000e+01
All other variables are zero.
```

Interpretation

- Jan: Issue commercial paper for CHF 150k
- Feb: Issue commercial paper for CHF 100k
- Mar: Issue paper for $\approx$CHF 152k and invest $\approx$ CHF 352k
- Apr: Take excess to pay outgoing cashflow
- May: Take a credit of CHF 52k
- Jun: Excess of $\approx$ CHF 92k

Dedication

Dedication is ..

.. technique to fund known liabilities in the future.
Intent is to form portfolio of assets whose cash inflow exactly offsets cash outflow of liabilities.
Used in small pension funds.
In our example, only risk-free assets (bonds) are used.
Optimal dedicated portfolio is computed using linear programming.

Example

Liability schedule

Year 1	Year 2	Year 3	Year 4	Year 5	Year 6	Year 7	Year 8
12 000	18 000	20 000	20 000	16 000	15 000	12 000	10 000

Bonds

Bond	1	2	3	4	5	6	7	8	9	10
Prize	102	99	101	98	98	104	100	101	102	94
Coupon	5	3.5	5	3.5	4	9	6	8	9	7
Maturity year	1	2	2	3	4	5	5	6	7	8

Constants and Variables for LP

$i = 1, \dots, 10$, $t = 1, \dots, 8$

L_t: Liability in year t, p_i: Prize of bond i, c_i: Annual coupon for bond i, m_i maturity year of bond i

x_i: Amount of bond i in portfolio, z_t: surplus at end of year t, z_0: surplus at year 0

Setting up a linear program

Objective: $\min z_0 + \sum_{i=1}^{10} p_i x_i$

$\forall 1 \leqslant t \leqslant 8: \quad \sum_{i:m_i \geqslant t} c_i x_i + \sum_{i:m_i = t} 100 \cdot x_i - z_t + z_{t-1} = L_t$

Exercise (will appear on the next exercise sheet)

Write a zimpl-model for the dedication-problem above and compute the optimal solution using Soplex.

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Part 3

The fundamental Theorem of Asset Pricing

Primary objectives:

- Arbitrage: Definition and example
- Duality and complementary slackness
- Fundamental theorem of asset pricing
- Arbitrage detection using linear programming

Arbitrage

Arbitrage is trading strategy that:

1. has positive initial cash flow and no risk of loss later (Type A)
2. requires no initial cash input, has no risk of loss and has positive probability of making profits in the future (Type B)

Call Options

European call option

- At expiration date the holder has right to
- purchase prescribed asset underlying
- for prescribed amount

Call Options

Pricing derivative security

- S_0 current price of underlying security
- Two possible outcomes up and down at expiration date:

$$S_1^u = S_0 \cdot u$$
$$S_1^d = S_0 \cdot d$$

- How to price the derivative security?

Replication

- Consider portfolio of $\triangle$ shares of the underlying and B cash
- Up-state: $\triangle \cdot S_0 \cdot u + B \cdot R$, ($R$ is risk-less interest rate)
- Down-state: $\triangle \cdot S_0 \cdot d + B \cdot R$
- For what values of $\triangle$ and B will portfolio have same payoff C_1^u and C_1^d of derivate?

Replication

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$$\begin{aligned}
\triangle \cdot S_0 \cdot u + B \cdot R &= C_1^u = \max\{u \cdot S_0 - C_0, 0\} \\
\triangle \cdot S_0 \cdot d + B \cdot R &= C_1^d = \max\{d \cdot S_0 - C_0, 0\}
\end{aligned}$$

Replication

- Consider portfolio of $\triangle$ shares of the underlying and B cash
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$$\begin{aligned} \triangle \cdot S_0 \cdot u + B \cdot R &= C_1^u = \max\{u \cdot S_0 - C_0, 0\} \\ \triangle \cdot S_0 \cdot d + B \cdot R &= C_1^d = \max\{d \cdot S_0 - C_0, 0\} \end{aligned}$$

- One obtains

$$\triangle = \frac{C_1^u - C_1^d}{S_0(u-d)}$$
$$B = \frac{uC_1^d - dC_1^u}{R(u-d)}$$

- Since portfolio is worth $S_0\triangle + B$ today, this should also be price for derivate security

$$
\begin{aligned}
C_0 &= \frac{C_1^u - C_1^d}{u-d} + \frac{uC_1^d - dC_1^u}{R(u-d)} \\
&= \frac{1}{R}\left[\frac{R-d}{u-d}C_1^u + \frac{u-R}{u-d}C_1^d\right]
\end{aligned}
$$

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Risk-neutral probabilities

$p_u = \frac{R-d}{u-d}$, $p_d = \frac{u-R}{u-d}$

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\end{aligned}
$$

Risk-neutral probabilities

$p_u = \frac{R-d}{u-d}$, $p_d = \frac{u-R}{u-d}$

Remark

If price $\neq C_0$, then there is arbitrage opportunity.

Complementary slackness

Recall LP Duality

Let $\min\{c^T x \mid Ax = b, x \geqslant 0\}$ be an LP in standard form. If LP is feasible and bounded, then also the dual LP $\max\{b^T y \mid A^T y \leqslant c\}$ is feasible and bounded and there exist optimal solutions x^* and y^* of the primal resp. dual with $c^T x^* = b^T y^*$

Complementary Slackness

Let x^* and y^* be feasible solutions of the primal and dual linear program. The following conditions are equivalent:

1. x^* and y^* are optimal solutions of primal and dual respectively.
2. $x^*(i) > 0 \Longrightarrow (c - A^T y^*)(i) = 0$

Arbitrage

Arbitrage is trading strategy that:

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Generalization of binomial (2-stage) setting

- Let $\omega_1, \dots, \omega_m$ be finite set of possible states
- For securities S^i, $i = 0, \dots, n$ let $S^i_1(\omega_j)$ be price of security in state ω_j at time 1 and let S^i_o be price of security at time 0
- S^0 is riskless security that pays interest rate $r \geqslant 0$ between time 0 and time 1; $S^0_0 = 1$ and $S^0_1(\omega_j) = R = (1+r)$ for $j = 1, \dots, m$

Risk-neutral probability

A risk-neutral probability measure on the set $\Omega = \{\omega_1, \dots, \omega_m\}$ is a vector $p_1, \dots, p_m$ of positive numbers with

$$\sum_{j=1}^{m} p_j = 1$$

and for every S^i, $i = 0, \dots, n$ one has

$$S^i_0 = \frac{1}{R} \sum_{j=1}^{m} p_j S^i_1(\omega_j) = \frac{1}{R} E[S^i_1].$$

Fundamental theorem of asset pricing

First fundamental theorem of asset pricing

A risk-neutral probability measure exists if and only if there is no arbitrage

Summary of covered asset pricing topics

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