Optimization Methods in Finance

Welcome!

- My name is Friedrich Eisenbrand
- Professor of Mathematics at EPFL
- Research interests: Discrete Optimization, Algorithms and Applied Optimization
- Assistant of the course: Thomas Rothvoß

How to contact me:

- Come to see me during office ours: Tuesdays 11-12
- Send me an e-mail `friedrich.eisenbrand@epfl.ch` at least one day ahead, announcing your visit
- Apart from personal contact: I do not guarantee to answer your e-mails!
- **Highly appreciated:** E-mails about errors/typo's on my slides

Course webpage

See `http://disopt.epfl.ch` and follow the Teaching link!

Syllabus

Topics

- Linear Programming: Simplex Method, Computing a dedicated bond portfolio, asset pricing
- Quadratic Programming: Portfolio Optimization (Markowitz model)
- Integer Programming: Constructing an index fund
- Dynamic Programming: Option Pricing, Structuring asset backed securities
- Stochastic Programming: Asset/Liability management

Prerequisites and literature

Prerequisites

- Linear Algebra
- Basic knowledge in Algorithms: O-Notation, Pseudocode, …
- Basic probability theory

Literature

- G. Cornuéjols and R. Tütüncü, Optimization Methods in Finance, Cambridge (main reference)
- V. Chvatal, Linear Programming, Freeman & Company
- D. Bertsimas & J.N. Tsitsiklis, Introduction to Linear Optimization, Athena Scientific

Main skills

- Learning of basic optimization methods and how to apply them in the world of finance: Modeling, practical problem solving (case studies) using software packages
- Mathematical rigor: Proofs, analysis of running time, correctness of methods

Organization

Exercises

- Every two weeks there is an assignment sheet
- First assignment is available in the 2nd week
- In the tutorial between two releases, the students may ask questions concerning the current exercises (or the lecture)
- In the tutorial 2 weeks after release of the sheet, the exercises are discussed; students will present the solutions
- There is neither a submission nor a correction of the exercise sheets

Grading

- Final Grade: 30% Midterm, 30% Oral presentation of practical exercises (1 or 2 larger case studies), 40% Final exam.
- Midterm and Final are written exams, 120 minutes each

Part 1
Linear Programming

Notation

Let $A \in \mathbb{R}^{m\times n}$, $v \in \mathbb{R}^n$ and $J = \{j_1, \ldots, j_k\} \subseteq \{1, .., n\}$ with $j_1 \leqslant j_2 \leqslant \ldots \leqslant j_k$.

- For $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$:
  - a^j: j-th column of A
  - a_i: i-th row of A
  - $A(i, j)$: Element of A in i-th row and j-th column
  - $v(i)$ (also v_i): i-th component of v
- A_J: Matrix $\left(a^{j_1}, \ldots, a^{j_k}\right)$
- v_J: Vector $(v(j_1), \ldots, v(j_k))^T$

Definition: Linear Program

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, $I_\geqslant, I_\leqslant, I_= \subseteq \{1, \ldots, m\}$ and $J_\geqslant, J_\leqslant \subseteq \{1, \ldots, n\}$
Linear Program (LP) consists of:

- Objective function:

$$\max c^T x$$
$$\text{or } \min c^T x$$

- Constraints

$$a_i^T x \geqslant b(i), i \in I_\geqslant$$
$$a_j^T x \leqslant b(j), j \in I_\leqslant$$
$$a_k^T x = b(k), k \in I_=$$

- Bounds on Variables

$$x_{J_\geqslant} \geqslant 0$$
$$x_{J_\leqslant} \leqslant 0$$

Example: Modelling Cashflow Management

Our company has net cash flow requirements (in 1000 CHF)

Month	Jan	Feb	Mar	April	May	June
Net cash flow	-150	-100	200	-200	50	300

For example in January we have to pay CHF 150k, while in March we will get CHF 300k. Initially we have 0 CHF, but the following possibilities to borrow/invest money

1. We have a credit line of CHF 100k per month. For a credit we have to pay 1% interest per month.
2. In the first 3 months we may issue a 90-days commercial paper (to borrow money) at a total interest of 2% (for the whole 3 months).
3. Excess money can be invested at an interest rate of 0.3% per month.

Task: We want to maximize our wealth in June, while we have to fulfill all payments. How shall we invest/borrow money?

Setting up a Linear Program

Introduce decision variables

$$\begin{aligned} x_i &= \text{credit in month } i \in \{1,\dots,6\} \\ y_i &= \text{amount of money due to commercial paper } i \in \{1,2,3\} \\ z_i &= \text{fund excess in month } i \in \{1,\dots,6\} \end{aligned}$$

Define $b = (-150, -100, 200, -200, 50, 300)$ as the vector of net cash flow. Then the following LP determines the optimal investment strategy:

$$\begin{aligned} \max z_6 - x_6 & & & \\ x_1 + y_1 - z_1 &= -b_1 & & \\ x_i + y_i - 1.01x_{i-1} + 1.003z_{i-1} - z_i &= -b_i & & \forall i = 2,3 \\ x_i - 1.02y_{i-3} - 1.01x_{i-1} + 1.003z_{i-1} - z_i &= -b_i & & \forall i = 4,\dots,6 \\ 0 \leqslant x_i &\leqslant 100 & & \forall i = 1,\dots,6 \\ z_i &\geqslant 0 & & \forall i = 1,\dots,6 \\ y_i &\geqslant 0 & & \forall i = 1,2,3 \end{aligned}$$

Software

ZIMPLE:

- A modelling language for linear/integer programs
- Enables compact formulations of optimization problems
- Command `zimpl` converts `zpl` files to `lp` or `mps` files which can be solved with most solvers
- Free software for Windows/Linux/MacOS
- Website: `http://zimpl.zib.de/`
- Good documentation available

QSOpt:

- LP/IP solver for Windows/Linux/MacOS
- Can solve problems, described as `lp` or `mps` file
- Website: `http://www2.isye.gatech.edu/~wcook/qsopt/`

Modelling with ZIMPL

File: cashflow.zpl

```
set Months := {1,2,3,4,5,6};
param b[Months] :=< 1 > -150, < 2 > -100, < 3 > 200,
                  < 4 > -200, < 5 > 50, < 6 > 300;

var x[Months] real >= 0 <= 100;
var y[{1,2,3}] real >= 0;
var z[Months] real >= 0;

maximize wealth : z[6] - x[6];
subto Jan : x[1] + y[1] - z[1] == -b[1];
subto FebToMar : forall < i >  in {2..3} do
 x[i] + y[i] - 1.01 * x[i - 1] + 1.003 * z[i - 1] - z[i] == -b[i];
subto AprToJun : forall < i > in{4..6} do
 x[i] - 1.02 * y[i - 3] - 1.01 * x[i - 1] + 1.003 * z[i - 1] - z[i] == -b[i];
```

Running `zimple cashflow.zpl` creates `cashflow.lp`

The LP File

File `cashflow.lp`

```
\This file was automatically generated by Zimpl
...
Maximize
 wealth: -x#6 + z#6
Subject to
Jan_1:   -z#1 + y#1 + x#1 = 150
FebToMar_1: -z#2 + 1.003z#1 - 1.01x#1 + y#2 + x#2 = 100
FebToMar_2: -z#3 + 1.003z#2 - 1.01x#2 + y#3 + x#3 = -200
AprToJun_1: -z#4 + 1.003z#3 - 1.01x#3 - 1.02y#1 + x#4 = 200
AprToJun_2: -z#5 + 1.003z#4 - 1.01x#4 - 1.02y#2 + x#5 = -50
AprToJun_3: -z#6 + 1.003z#5 - 1.01x#5 - 1.02y#3 + x#6 = -300
Bounds
0 <= x#1 <= 100
...
0 <= z#6 <= +inf
End
```

Solving the LP File with QSOpt

Results of `qsopt -O cashflow.lp`

```
LP Value: 92.496949
Time for SOLVER: 0.01 seconds.
Solution Values
z#6 = 92.496949
y#1 = 150.000000
y#2 = 100.000000
z#3 = 351.944167
y#3 = 151.944167
x#5 = 52.000000
```

Interpretation:

- Jan: Issue commercial paper for CHF 150k
- Feb: Issue commercial paper for CHF 100k
- Mar: Issue paper for ≈CHF 152k and invest ≈ CHF 352k
- Apr: Take excess to pay outgoing cashflow
- May: Take a credit of CHF 52k
- Jun: Excess of ≈ CHF 92k

Definitions

feasible LP, bounded LP

- $x^* \in \mathbb{R}^n$ is feasible of feasible solution of LP, if x^* satisfies all constraints and bounds on variables
- LP is feasible if there exist feasible solutions of LP. Otherwise LP is infeasible
- LP is bounded, if LP is feasible and there exits constant $M \in \mathbb{R}$ with $c^T x < M$ for all feasible solutions $x \in \mathbb{R}^n$ if LP is a maximization problem or $c^T x > M$ for all feasible $x \in \mathbb{R}^n$ if LP is minimization problem

Transformations

1) $\max c^T x$ can be replaced by $\min -c^T x$
2) $a_i^T x \leqslant b(i)$ can be replaced by $-a_i^T x \geqslant -b(i)$
3) Bound $x(j) \geqslant 0$ or $x(j) \leqslant 0$ can be replaced by constraint $e_j^T x \geqslant 0$ or $e_j^T x \leqslant 0$ respectively, where e_j is j-th unit vector
4) $a_i^T x \leqslant b(i)$ can be replaced by $a_i^T x + s = b(i)$, where s is a new variable which is bounded by $s \geqslant 0$
5) Conversion of an unbounded variable $x(j)$ in a bounded variable: Introduce new bounded variables $p_+(j), p_-(j) \geqslant 0$ and replace each occurrence of $x(j)$ by $p_+(j) - p_-(j)$
6) Conversion of a variable $x(j)$ bounded by $\leqslant 0$ into a variable which is bounded by $\geqslant 0$: Replace bound $x(j) \leqslant 0$ by $x(j) \geqslant 0$ and replace each other occurrence of $x(j)$ by $-x(j)$ (also in objective function!)

Standard forms

- Inequality standard form :

$$\max c^T x$$
$$Ax \leqslant b$$

- Equality standard form or simply standard form :

$$\min c^T x$$
$$Ax = b$$
$$x \geqslant 0$$

Example

$$\max 2\cdot x(1) - 4\cdot x(2)$$
$$3\cdot x(1) + 2\cdot x(2) \geqslant 4$$
$$2\cdot x(1) + 4\cdot x(2) \geqslant 9$$
$$x(2) \leqslant 0$$

is to be transformed into standard form.
Application of 1) and 2)

$$\min -2\cdot x(1) + 4\cdot x(2)$$
$$-3\cdot x(1) - 2\cdot x(2) \leqslant -4$$
$$-2\cdot x(1) - 4\cdot x(2) \leqslant -9$$
$$x(2) \leqslant 0$$

Application of 6)

$$\min -2 \cdot x(1) - 4 \cdot x(2)$$
$$-3 \cdot x(1) + 2 \cdot x(2) \leqslant -4$$
$$-2 \cdot x(1) + 4 \cdot x(2) \leqslant -9$$
$$x(2) \geqslant 0$$

Application of 5)

$$\min -2 \cdot p_+(1) + 2 \cdot p_-(1) - 4 \cdot x(2)$$
$$-3 \cdot p_+(1) + 3 \cdot p_-(1) + 2 \cdot x(2) \leqslant -4$$
$$-2 \cdot p_+(1) + 2 \cdot p_-(1) + 4 \cdot x(2) \leqslant -9$$
$$p_+(1), p_-(1), x(2) \geqslant 0$$

Application of 4)

$$\min -2 \cdot p_+(1) + 2 \cdot p_-(1) - 4 \cdot x(2) + 0 \cdot s(1) + 0 \cdot s(2)$$
$$-3 \cdot p_+(1) + 3 \cdot p_-(1) + 2 \cdot x(2) + s(1) = -4$$
$$-2 \cdot p_+(1) + 2 \cdot p_-(1) + 4 \cdot x(2) + s(2) = -9$$
$$p_+(1), p_-(1), x(2), s(1), s(2) \geqslant 0$$

Convention

In the following, we will always assume LPs to be in standard form

$$\begin{aligned} \min c^T x \\ Ax = b \\ x \geqslant 0 \end{aligned} \tag{1}$$

Reminder

- Vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ are linearly independent, if one has $\sum_{i=1}^k x(i)\, v_i \neq 0$ for all $x \neq 0\ \mathbb{R}^k$
- Column-rank of A is maximal number of linearly independent columns of A
- Row-rank of A is maximal number of linearly independent rows of A
- Rank of A, $\text{rank}(A)$: Maximal number of linearly independent columns of rows of A
- A has full row-rank (column-rank), if $\text{rank}(A) = m$ ($\text{rank}(A) = n$)

W.l.o.g. A has full row-rank

- Suppose $\operatorname{rank}(A) < m$ and let the first row be in the span of the other rows

$$a_1 = \sum_{i=2}^{m} \lambda_i a_i \text{ with suitable numbers } \lambda_2, \ldots, \lambda_m \in \mathbb{R}.$$

- If $\sum_{i=2}^{m} \lambda_i b(i) = b(1)$, then one has for all $x \in \mathbb{R}^n$ with $a_i^T x = b(i), i = 2, \ldots, m$ also $a_1^T x = b(1)$ which means that the first equation in $Ax = b$ can be discarded.
- If $\sum_{i=2}^{m} \lambda_i b(i) \neq b(1)$, then there does not exist an $x \in \mathbb{R}^n$ with $Ax = b$ and the LP (1) is infeasible.

Convention

In the following we will assume that A has full row-rank.

Lemma 1.1

If LP (1) has an optimal solution, then there exists an optimal solution $x^* \in \mathbb{R}^n$ of (1) such that A_J has full column rank, where $J = \{j \mid x^*(j) > 0\}$.

Proof

- Let x^* be an optimal solution and suppose that columns of A_J are linearly dependent
- Idea: Compute new optimal solution $\tilde{x}$ with $J' = \{j \mid \tilde{x}(j) > 0\} \subset J$
- After finite number of repetitions of this step, one has optimal solution which satisfies the condition of the theorem
- Let $\overline{J} = \{1, \ldots, n\} \setminus J$
- A_J does not have full column rank $\Longrightarrow \exists\, d \in \mathbb{R}^n$ with $d \neq 0$, $Ad = 0$ and $d(j) = 0\ \forall j \in \overline{J}$
- $\Longrightarrow x^* \pm \varepsilon d$ is feasible for $\varepsilon > 0$ sufficiently small.

Proof cont.

- $c^T(x^* \pm \varepsilon d) = c^T x^* \pm \varepsilon c^T d$; since x^* is optimal it follows that $c^T d = 0$
- Via eventually changing d to $-d$ we can assume that there exists a $j \in J$ with $d(j) < 0$
- Consider $x^* + \varepsilon d$. Goal: Choose $\varepsilon > 0$ in such a way that $x^* + \varepsilon d$ still feasible ($\iff x^* + \varepsilon d \geqslant 0$) but also $(x^* + \varepsilon d)(j) = 0$ for some $j \in J$
- How large can we choose $\varepsilon > 0$ without getting infeasible?
- Infeasibility can be caused by indices j with $d(j) < 0$; Let $K \subseteq J$ be the set of these indices
- We need for all $k \in K$ the condition:

$$\begin{aligned} & x^*(k) + \varepsilon\, d(k) \geqslant 0 \\ \iff \quad & \varepsilon \leqslant -x^*(k)/d(k) \end{aligned}$$

Proof cont.

- Let $k' \in K$ be an index with

$$-x^*(k')/d(k') = \min_{k \in K} -x^*(k)/d(k)$$

  and let $\varepsilon' = -x^*(k')/d(k')$
- Then $x^* + \varepsilon' d$ is feasible and $J' = \{j \mid (x^* + \varepsilon' d)(j) > 0\} \subset J$. □

Definitions

Basic solution, basic feasible solution, associated basis

- $x^* \in \mathbb{R}^n$ is called a basic solution, if $Ax^* = b$ and $\text{rank}(A_J) = |J|$, where $J = \{j \mid x^*(j) \neq 0\}$; A basic solution is a feasible basic solution, if $x^* \geqslant 0$
- Basis is index set $B \subseteq \{1, \ldots, n\}$ with $\text{rank}(A_B) = m$ and $|B| = m$
- $x^* \in \mathbb{R}^n$ with $A_B x_B^* = b$ and $x^*(j) = 0$ for all $j \notin B$ is basic solution associated to basis B.

Lemma 1.2

Each basic solution x^* is associated to (at least) one basis B

Proof

- Let $J = \{j \mid x^*(j) \neq 0\}$
- Columns of A_J are linearly independent
- Augment J to index set $B \supseteq J$ such that A_B is invertible
- One has $A_B x^*_B = b$ and $x^*(j) = 0$ for all $j \notin B$

Exercise

Show that a basic solution can be associated to two different bases.

A naive algorithm for linear programming

Let $\min\{c^T x \mid Ax = b, x \geqslant 0\}$ be a bounded LP

- Enumerate all bases $B \subseteq \{1, \ldots, n\}$ $\mathbf{O}(\binom{n}{m}) = \mathbf{O}(\mathbf{n^m})$ many
- Compute associated basic solution x^* with $x^*_B = A_B^{-1} b$ and $x^*_{\overline{B}} = 0$
- Return the one which has largest objective function value among the feasible basic solutions
- Running time $O(n^m \cdot m^3)$
- **Are there more efficient algorithms?**