Today: Elementary Prime number estimates.

<u>Thm:</u> There exist positive constants $C_1, C_2 > 0$ such that for all $x \geq 2$ one has

$$\frac{C_1 \cdot x}{\log(x)} \leq \pi(x) \leq \frac{C_2 \cdot x}{\log(x)}$$

<u>Lemma:</u> For every $x \geq 0$ one has

$$\prod_{\substack{p \leq x \\ p \in \mathbb{P}}} p \leq 4^x$$

<u>Proof of upper bound:</u>

$$4^x \geq \prod_{\substack{p \leq x \\ p \in \mathbb{P}}} p \geq \prod_{\substack{\sqrt{x} \leq p \leq x \\ p \in \mathbb{P}}} p \geq \sqrt{x}^{\,\pi(x) - \pi(\sqrt{x})}$$

$$\Rightarrow \left[\pi(x) - \pi(\sqrt{x})\right] \cdot \log(\sqrt{x}) \leq x \cdot \log 4$$

$$\Rightarrow \pi(x) \leq \frac{x \cdot \log 4}{\frac{\log(x)}{2}} + \pi(\sqrt{x})$$

$$\leq \frac{2 \cdot \log 4 \, x}{\log x} + \sqrt{x}$$

$$\leq C_2 \cdot \frac{x}{\log x}$$

$\blacksquare$

Proof of Lemma:

We can assume that $x ~~\in~~ . \in \mathbb{N}_0$.

Induction: $n=0$, $n=1$ $\quad \prod_{\substack{p \le 0 \\ p \in \mathbb{P}}} p = 1 = 4^0$

$n=2$: $\quad \prod_{\substack{p \le 2 \\ p \in \mathbb{P}}} p = 2 \le 4^2$

~~[illegible]~~

$n > 2$

If n even: $\quad \prod_{\substack{p \le n \\ p \in \mathbb{P}}} p = \prod_{p \le n-1} p \le 4^{n-1} \le 4^n$

If n odd: $\quad n = 2m+1$

Observation: if p is prime with $m+1 < p \le 2m+1$, then

$$p \mid \binom{2m+1}{m} = \frac{(2m+1)!}{(m+1)!\, m!}$$

Thus $\prod_{m+1 < p \le 2m+1} p \;\Big|\; \binom{2m+1}{m} \le 2^{2m}$

This means that

$$\prod_{p \le 2m+1} p \le 2^{2m} \cdot \prod_{p \le m+1} p$$

$$\le 2^{2m} \cdot 4^{m+1}$$

$$= 4^m \cdot 4^{m+1} = 4^{2m+1} \quad \blacksquare$$

<u>proof of lower bound:</u> $\quad c_1 \cdot \frac{x}{\log(x)} \leq \pi(x) \quad , x \geq 2 .$

<u>Lemma:</u> $0 \leq i \leq n$ integers, then every prime power dividing $\binom{n}{i}$ is $\leq n$.

$$\left[p^{\alpha} \mid \binom{n}{i} \Rightarrow p^{\alpha} \leq n \right].$$

proof later.

$$\binom{n}{i} = \prod_{p \leq n} p^{\alpha(p)} \leq \prod_{p \leq n} n = n^{\pi(n)}$$

$$2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} = (1+1) + \sum_{i=1}^{(n-1)} \binom{n}{i}$$

$$\leq 2 + (n-1) \cdot n^{\pi(n)}$$

$$\leq n^{\pi(n)+1}$$

$$\Rightarrow \pi(n) + 1 \geq \frac{n}{\log n}$$

Proof of Lemma:

$p \quad 2p \quad 3p \quad pp$

$1 \cdot 2 \cdot 3 \cdots\cdots \cdot n$

$$\mathrm{ord}_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor$$

$$\mathrm{ord}_p(n!) = \#(\text{multiples of } p \leq n)$$

$$+ \#(\text{multiples of } p^2 \leq n)$$

$$\vdots$$

$$= \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor$$

Now

$$\mathrm{ord}_p \binom{n}{a} = \mathrm{ord}_p \frac{n!}{a!\,(n-a)!}$$

$$= \sum_{i=1}^{\infty} \left(\left\lfloor \frac{n}{p^i} \right\rfloor - \cancel{\sum_{i=1}^{\infty}} \left\lfloor \frac{a}{p^i} \right\rfloor - \left\lfloor \frac{n-a}{p^i} \right\rfloor \right)$$

$$\leq \log_p n$$

$$\Rightarrow p^{\mathrm{ord}_p \binom{n}{a}} \leq p^{\log_p n} = n.$$