Algorithms for Market Equilibria:

Given: B: set of n' buyers

A: set of n divisible goods

e_i: amount of money available to buyer i

b_j: " goods of type j

u_{ij}: utility of good j to buyer i

happyness of buyer: i buys x_{ij} units of j

$$\text{happyness}(i) := \sum_{j=1}^{n} u_{ij} x_{ij} \quad ,$$

Optimal basket of goods for buyer i:

given by LP:

$$\max \sum_{j=1}^{n} u_{ij} x_{ij}$$

$$\text{s.th.} \quad \sum_{j=1}^{n} p_j \cdot x_{ij} \le e_i$$

$$x_{ij} \le b_j \qquad \forall j = 1, \dots, n$$

$$x \ge 0 \quad ,$$

where p_j is price of good j.

$p_1, \dots, p_n$ are market cleaning prices if,

after each buyer buys his optimal basket of goods, then

- there are no goods left
- there is no deficiency of goods.

I.o.w: $\forall j: \sum_{i=1}^{n'} x_{ij} = b_j$

Goal of todays lecture: Show that market equilibria (market cleaning) prices exist, and that those can be efficiently computed.

Need Theory of Duality for (convex) programs

Consider:
$$\min f_0(x) \qquad \text{s.th. } f_i(x) \leq 0, \quad i = 1, \dots, m \qquad (I)$$
with domain $D \subseteq \mathbb{R}^n$

Lagrangean:
$$L(x, \lambda) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x)$$

$$g(\lambda) = \inf_{x \in D} L(x, \lambda)$$

Thm: (Weak Duality)

Suppose (I) has optimal solution with opt. value p^*, then $\forall \lambda \geq 0: \quad g(\lambda) \leq p^*$

Proof: Suppose $\bar{x}$ is feasible i.e. $\bar{x} \in D$ and

$$f_i(\bar{x}) \leq 0 \quad, i = 1, \dots, m. \quad \text{Let } \lambda \geq 0, \text{ then}$$

$$L(\bar{x}, \lambda) = f_0(\bar{x}) + \underbrace{\sum_{i=1}^{m} \underbrace{\lambda_i}_{\geq 0} \underbrace{f_i(\bar{x})}_{\leq 0}}_{\leq 0}$$

$$\leq f_0(\bar{x}).$$

This implies $g(\lambda) \leq p^*$ for each $\lambda \geq 0$.

∎

Lagrange Dual Problem:

$$\max \; g(\lambda)$$
$$\text{s.t. } \lambda \geq 0$$

Observation:

$g(\lambda)$ is concave

$\theta \in [0,1]$, $\lambda_1, \lambda_2 \geq 0$, $x \in D$:

$$L(x, \theta \cdot \lambda_1 + (1-\theta)\lambda_2) = f_0(x) + \theta \sum_{i=1}^{m} \lambda_i^1 f_i(x) + (1-\theta) \sum_{i=1}^{m} \lambda_i^2 f_i(x)$$

$$= \theta L(x, \lambda_1) + (1-\theta) L(x, \lambda_2)$$

$$\Rightarrow g(\theta \cdot \lambda_1 + (1-\theta) \cdot \lambda_2) \geq \theta \cdot g(\lambda_1) + (1-\theta) g(\lambda_2)$$

$\Rightarrow$ Finding $\max\limits_{\lambda \geq 0} g(\lambda)$ is convex program,

regardless of whether (I) is convex program

Slater's condition: (baby version)

If D is full-dimensional and $f_0, f_1, \dots, f_m$ are convex, then (I) satisfies Slater's condition.

Remark: More general version can be found in book: Convex Optimization; Boyd, Vanderberghe, P 226.

Thm: If (I) satisfies Slater's condition ~~then~~ and (I) is feasible, then

$$\max_{\lambda \geq 0} g(\lambda) = \min f_0(x) = p^*$$
$$\text{s.th. } f_i(x) \leq 0$$

proof:

Define $A = \{ (u, t) : \exists x \in D \text{ with } f_i(x) \leq u_i, i = 1, \dots, m$ and $f_0(x) \leq t \}$

A is convex. Let $\bar{x} \in D$ satisfy $f_i(\bar{x}) < 0 \quad i = 1, \dots, m$

Define $B = \{ (0, s) : s < p^* \}$

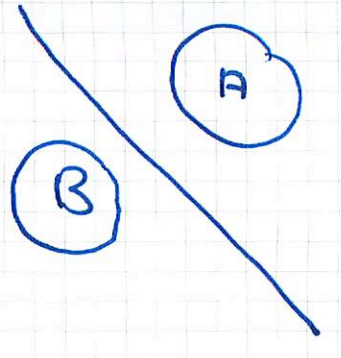
$A \cap B = \emptyset$ since if $(0, s) \in B$ and $(0, s) \in A$, then

$\exists x \in D$ with $f_i(x) \leq 0 \quad i = 1, \dots, m$ and

$f_0(x) \leq s < p^*$ ↯.

Separating hyperplane thm $\Rightarrow$

$\exists (\lambda, \mu) \neq 0$ and $\alpha \in \mathbb{R}$ s.th.



$(u, t) \in A \Rightarrow \lambda^T \cdot u + \mu_0 t \geq \alpha \quad (*)$

$(u, t) \in B \Rightarrow \lambda^T \cdot u + \mu \cdot t \leq \alpha \quad (**)$

$\lambda \geq 0$ and $\mu \geq 0$, otherwise $\lambda^T \cdot u + \mu \cdot t$ is unbounded from below if evaluated at A.

$(**) \Rightarrow \mu \cdot t \leq \alpha \quad \forall\, t < p^*$ and thus $\mu \cdot p^* \leq \alpha$

together with $(*)$: $\quad \forall x \in D$:

$$\sum_{i=1}^{m} \lambda_i f_i(x) + \mu \cdot f_0(x) \geq \alpha \geq \mu \cdot p^*$$

if $\mu > 0$:

$$\sum_{i=1}^{m} \frac{\lambda_i}{\mu} f_i(x) + f_0(x) \geq p^*$$

$$\Rightarrow g\left(\frac{\lambda}{\mu}\right) \geq p^*.$$

if $\mu = 0$: $\quad \sum_{i=1}^{m} \lambda_i f_i(x) \geq 0 \quad \forall x \in D.$

in particular for $\bar{x}$: $\quad \sum_{i=1}^{m} \lambda_i \underbrace{f_i(\bar{x})}_{<0} \geq 0$

$\Rightarrow \lambda = 0$ ↯.

∎

KKT Optimality Conditions

Suppose $f_i(x)$, $i = 0, \dots, m$ are differentiable but not necessarily convex.

Let x^*, λ^* be optimal primal and dual solutions with zero duality gap, i.e.

x^* minimizes $L(x, \lambda^*)$ over x. Thus gradient must vanish at x^*:

$$\nabla f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*) = 0$$

Thus we have:

$$\left\{ \begin{array}{ll} f_i(x^*) \leq 0 & i = 1, \dots, m \\ \lambda_i^* \geq 0 & i = 1, \dots, m \\ \lambda_i^* f_i(x^*) = 0 & i = 1, \dots, m \\ \nabla f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*) = 0 & \end{array} \right.$$

$\rightarrow$ KKT Conditions.

Back to Market Equilibria:

Eisenberg-Gale convex program:

Assume $b_i = 1$ $\forall i$ by scaling u_{ij}'s.

$$\min \sum_{i=1}^{n'} e_i \cdot \log\left(\sum_{j=1}^{n} u_{ij} x_{ij}\right)$$

s.th.

$$\sum_{i=1}^{n'} x_{ij} \leq 1 \qquad \forall j \in A$$

$$x_{ij} \geq 0 \qquad \forall i \in B, \forall j \in B.$$

Satisfies Slater's condition.

KKT conditions $p_j ::=$ $j \in A$ vars of first constraint

set:

(i) $\forall j \in A$: $p_j \geq 0$

(ii) $\forall j \in A$: $p_j > 0 \Rightarrow \sum_{i \in A} x_{ij} = 1$

(iii) $\forall i \in B, \forall j \in A$: $\dfrac{u_{ij}}{p_j} \leq \dfrac{\sum_{j \in A} u_{ij} x_{ij}}{e_i}$

(iv) $\forall i \in B$ $\forall j \in A$: $x_{ij} > 0 \Rightarrow \dfrac{u_{ij}}{p_j} = \dfrac{\sum_{j \in A} u_{ij} x_{ij}}{e_i}$

Theorem: If each good has potential buyer, then equilibrium exists.

$\forall j \ \exists i : u_{ij} > 0$

Proof:

$$p_j \geq \frac{e_i u_{ij}}{\sum_j u_{ij} x_{ij}} > 0 \qquad (iii)$$

by (ii): $\sum_{i \in A} x_{ij} = 1$.

$\Rightarrow$ all prices are strictly positive, all goods are sold.

iii) and iv): If i gets good j, then j is among goods that give buyer i max. utility per unit of money spent at current prices. Each buyer gets only a bundle ~~of~~ consisting of her most desired goods. i.e. an optimal bundle.

iv) equivalent to: $\forall i \in B, \forall j \in A$: $\dfrac{e_i u_{ij} x_{ij}}{\sum_{j \in A} u_{ij} x_{ij}} = p_j x_{ij}$

Summing over all j:

$$\forall i \in B \qquad \frac{e_i \quad \sum_j u_{ij} x_{ij}}{\sum_{j \in A} u_{ij} x_{ij}} = \sum_j p_j x_{ij}$$

$$\Rightarrow \forall i \in B: \; e_i = \sum_j p_j x_{ij}$$

$\Rightarrow$ money is fully spent, market clears.

$\blacksquare$