Introduction to Game Theory

Literature: V. Chvatal, Linear Programming (Chapter on games)

G. Owen, Game Theory (Nash equilibria)

Matrix Games:

$A \in \mathbb{R}^{m \times n}$ defines game for two players. Row-player chooses $i \in \{1, \dots, m\}$, Column player chooses $j \in \{1, \dots, n\}$.

$\left\{ \begin{matrix} \text{Payoff} \\ \text{Loss} \end{matrix} \right\}$ for $\left\{ \begin{matrix} \text{row-player} \\ \text{column-player} \end{matrix} \right\}$ is a_{ij}

Example: Each player ~~chooses~~ hides one or two coins and guesses the number of coins hidden by other player. If only one player makes right guess, then he receives a payoff being total number of hidden coins.

hide guess ↓ ↓	[1,1]	[1,2]	[2,1]	[2,2]
[1,1]	0	2	-3	0
[1,2]	-2	0	0	3
[2,1]	3	0	0	-4
[2,2]	0	-3	4	0

mixed strategy for $\begin{Bmatrix} \text{row} \\ \text{column} \end{Bmatrix}$ player is

$$\begin{Bmatrix} x \in \mathbb{R}^m_{\geq 0} \ , \ \sum_{i=1}^{m} x_i = 1 \\ y \in \mathbb{R}^n_{\geq 0} \ , \ \sum_{j=1}^{n} y_j = 1 \end{Bmatrix}$$

If row player stays with mixed strategy x, he assures himself an expected payoff of

$$\min_y \ x^T \cdot A \cdot y \qquad (I)$$

where y ranges over all mixed strategies of column player.

Lemma: $\min_y x^T \cdot A \cdot y = \min_j x^T \cdot A^j$

Notation:
A^j : j-th column
A_i : i-th row

proof: $\min x^T \cdot A \cdot y$

$\sum_{j=1}^{n} y_j = 1$ is LP in Standard form.

$y \geq 0$ Basic solutions are unit vectors.

$\square$

(I) is $\min_j x^T \cdot A^j$.

Row player wants to compute mixed strategy x^* s.th. x^* maximizes

$$\min x^T A^j$$

This is an LP:

$$\max \; z$$

$$\text{s.th.} \quad z - x^T \cdot A^j \leq 0, \qquad j = 1, \ldots, n$$

$$\sum_{i=1}^{m} x_i = 1$$

$$x \geq 0$$

Similarly, column player wants to compute y^* s.th.

$$\max_x \; x^T \cdot A \cdot y^* \quad \text{is minimized.}$$

This is equivalent to minimizing

$$\max_i \; A_i \, y$$

Again, this is an LP:

$$\min w$$

$$w - A_i \cdot y \geq 0, \quad i = 1, \dots, m$$

$$\sum_{j=1}^{n} y_j = 1$$

$$y \geq 0$$

The LP's are duals of each other. Both have feasible solutions.

Duality $\Rightarrow$ Opt. values are equal.

We have proved:

Thm: (Minimax Theorem)

$$\max_x \min_y x^T \cdot A \cdot y = \min_y \max_x x^T \cdot A \cdot y.$$ In other words,

there exist mixed strategies x^* and y^* such that

$$\min_y x^{*T} A \cdot y = \max_x x^T \cdot A \cdot y^*$$, where the y and x

in min and max respectively are ranging over all mixed strategies.

Two person Bi-Matrix Games and Nash's Theorem

$A, B \in \mathbb{R}^{m \times n}$ define payoffs for row and column player respectively.

A pair of mixed strategies x^* and y^* is Nash Equilibrium

if $\forall x: \quad x^T A y^* \leq x^{*T} A y^*$

$\forall y: \quad x^{*T} \cdot B \cdot y \leq x^{*T} \cdot B \cdot y^*$, where x and y are mixed strategies.

Thm: (Brower's Fix-Point Thm)
Let $C \subseteq \mathbb{R}^n$ be convex and compact and let $f: C \to C$ be a continuous function. There exists an element $s \in C$ with $f(s) = s$

Example: $f: [0,1] \to [0,1]$ continuous.

Consider $g(x) = f(x) - x$. We have $g(0) \geq 0$ and $g(1) \leq 0$.

An $x \in [0,1]$ with $g(x) = 0$ is a fix-point of f.

If $g(0) > 0$ and $g(1) < 0$, then $\exists\, x \in]0,1[$ with $g(x) = 0$

(Mean-value Theorem)

Thm (Nash)

Each bi-matrix game has a Nash equilibrium.