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An example for the Frank-Wolfe algorithm

Optimization Methods in Finance

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Consider the convex optimization problem

$$\begin{array}{lrcl} \min & x^T Qx & & \\ x_1 & +x_2 & \geq & 1 \\ x_1 & & \leq & 1 \\ & x_2 & \leq & 1 \end{array}$$

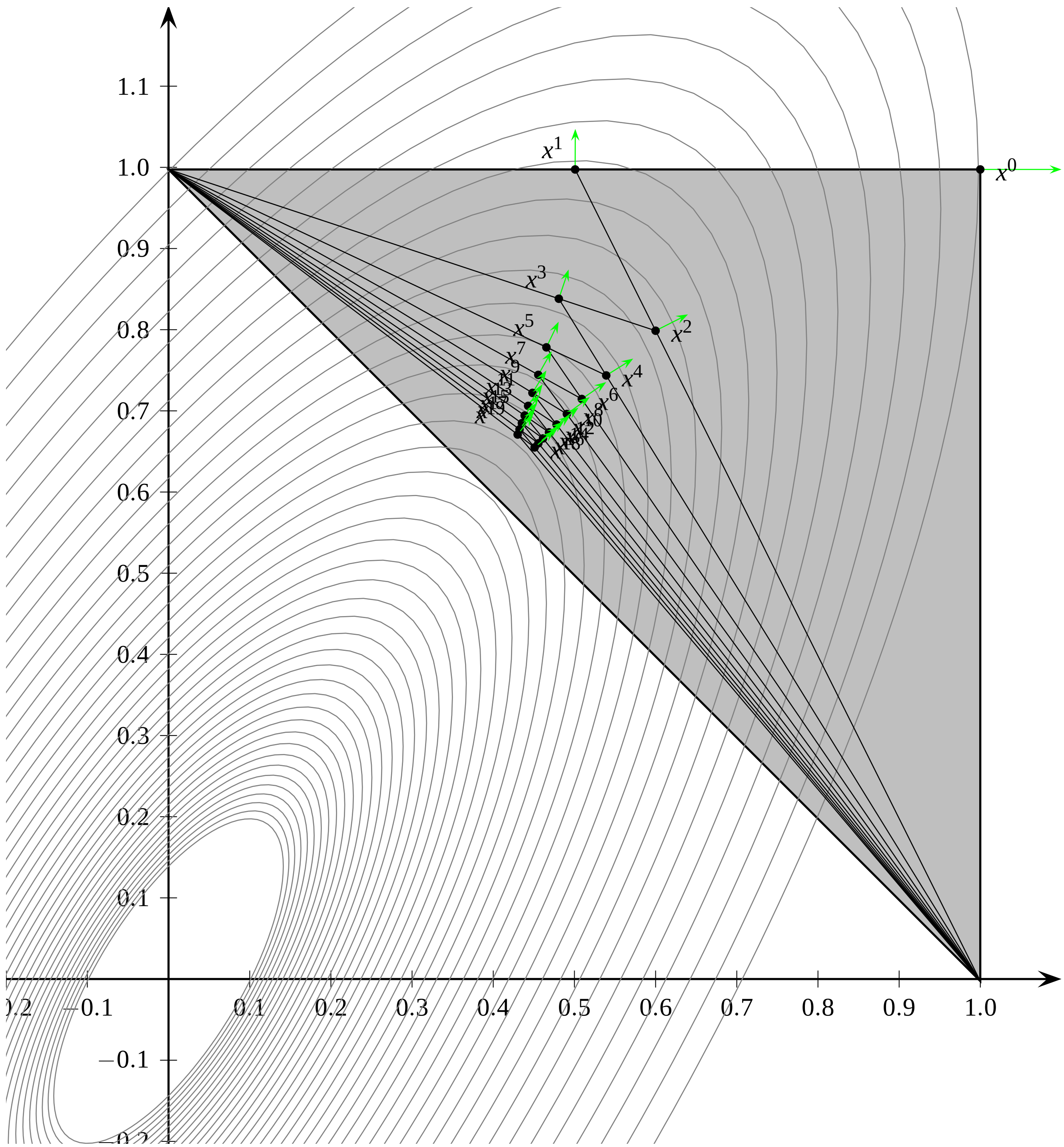
with

$$Q = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

Here Q is positive definite. We choose starting point $x^0 = (1,1)$ and abbreviate $f(x) = x^T Qx$. Then the Frank-Wolfe algorithm for 20 iterations performs as follows:

It	solution x^k	obj. value $f(x^k)$	gradient	y^k	λ^*
0	$x^0 = (1.0000, 1.0000)$	$f(x^0) = 1.0000$	$\nabla f(x^0) = (2.0000, 0.0000)$	$y^0 = (0,1)$	$\lambda^* = 0.4990$
1	$x^1 = (0.5009, 1.0000)$	$f(x^1) = 0.5000$	$\nabla f(x^1) = (0.0037, 0.9981)$	$y^1 = (1,0)$	$\lambda^* = 0.1986$
2	$x^2 = (0.6000, 0.8013)$	$f(x^2) = 0.4006$	$\nabla f(x^2) = (0.7977, 0.4024)$	$y^2 = (0,1)$	$\lambda^* = 0.1986$
3	$x^3 = (0.4808, 0.8407)$	$f(x^3) = 0.3607$	$\nabla f(x^3) = (0.2418, 0.7198)$	$y^3 = (1,0)$	$\lambda^* = 0.1124$
4	$x^4 = (0.5392, 0.7462)$	$f(x^4) = 0.3336$	$\nabla f(x^4) = (0.6645, 0.4139)$	$y^4 = (0,1)$	$\lambda^* = 0.1366$
5	$x^5 = (0.4655, 0.7808)$	$f(x^5) = 0.3161$	$\nabla f(x^5) = (0.3005, 0.6306)$	$y^5 = (1,0)$	$\lambda^* = 0.0815$
6	$x^6 = (0.5091, 0.7172)$	$f(x^6) = 0.3025$	$\nabla f(x^6) = (0.6022, 0.4160)$	$y^6 = (0,1)$	$\lambda^* = 0.1052$
7	$x^7 = (0.4555, 0.7469)$	$f(x^7) = 0.2924$	$\nabla f(x^7) = (0.3284, 0.5827)$	$y^7 = (1,0)$	$\lambda^* = 0.0645$
8	$x^8 = (0.4907, 0.6987)$	$f(x^8) = 0.2840$	$\nabla f(x^8) = (0.5652, 0.4161)$	$y^8 = (0,1)$	$\lambda^* = 0.0863$
9	$x^9 = (0.4483, 0.7247)$	$f(x^9) = 0.2774$	$\nabla f(x^9) = (0.3437, 0.5529)$	$y^9 = (1,0)$	$\lambda^* = 0.0538$
10	$x^{10} = (0.4780, 0.6857)$	$f(x^{10}) = 0.2716$	$\nabla f(x^{10}) = (0.5404, 0.4155)$	$y^{10} = (0,1)$	$\lambda^* = 0.0731$
11	$x^{11} = (0.4430, 0.7087)$	$f(x^{11}) = 0.2669$	$\nabla f(x^{11}) = (0.3545, 0.5315)$	$y^{11} = (1,0)$	$\lambda^* = 0.0461$
12	$x^{12} = (0.4687, 0.6761)$	$f(x^{12}) = 0.2627$	$\nabla f(x^{12}) = (0.5226, 0.4147)$	$y^{12} = (0,1)$	$\lambda^* = 0.0639$
13	$x^{13} = (0.4387, 0.6968)$	$f(x^{13}) = 0.2590$	$\nabla f(x^{13}) = (0.3613, 0.5161)$	$y^{13} = (1,0)$	$\lambda^* = 0.0403$
14	$x^{14} = (0.4613, 0.6686)$	$f(x^{14}) = 0.2558$	$\nabla f(x^{14}) = (0.5082, 0.4145)$	$y^{14} = (0,1)$	$\lambda^* = 0.0565$
15	$x^{15} = (0.4353, 0.6874)$	$f(x^{15}) = 0.2530$	$\nabla f(x^{15}) = (0.3663, 0.5042)$	$y^{15} = (1,0)$	$\lambda^* = 0.0360$
16	$x^{16} = (0.4556, 0.6626)$	$f(x^{16}) = 0.2504$	$\nabla f(x^{16}) = (0.4975, 0.4138)$	$y^{16} = (0,1)$	$\lambda^* = 0.0509$
17	$x^{17} = (0.4324, 0.6797)$	$f(x^{17}) = 0.2482$	$\nabla f(x^{17}) = (0.3703, 0.4946)$	$y^{17} = (1,0)$	$\lambda^* = 0.0326$
18	$x^{18} = (0.4510, 0.6575)$	$f(x^{18}) = 0.2460$	$\nabla f(x^{18}) = (0.4888, 0.4131)$	$y^{18} = (0,1)$	$\lambda^* = 0.0462$
19	$x^{19} = (0.4301, 0.6734)$	$f(x^{19}) = 0.2442$	$\nabla f(x^{19}) = (0.3737, 0.4865)$	$y^{19} = (1,0)$	$\lambda^* = 0.0298$

Below you can find a visualization of the algorithm. Gradients are depicted green (and downscaled to 5% of their real lengths). The gray ellipsoids depict some isopotentials $E_\beta = \{x \in \mathbb{R}^2 \mid x^T Qx = \beta\}$, where β grows in 10% steps.



1.1
1.0
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x^0
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