Two person general Sum Games

Two players: Row-Player and Column-Player

Two payoff matrices: $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m}$

x, y mixed strategies for ~~column~~ row and column player respectively, then

$x^T \cdot A \cdot y =$ Expected payoff for row-player.

$x^T \cdot B \cdot y =$ Expected payoff for column player.

Example: prisoners dilemma:

	Confess	Silent
Confess	4 / 4	1 / 5
Silent	5 / 1	2 / 2

(numbers are "negative" of payoff)

Only stable pure strategy is that both confess

Nash equilibrium: Pair x, y of mixed strategies s.th.

$\forall$ mixed strategies x': $\quad x'^T \cdot A \cdot y \leq x^T \cdot A \cdot y$

$\forall$ mixed strategies y': $\quad x^T \cdot B \cdot y' \leq x^T \cdot B \cdot y$

Lemma: Best response to column players (mixed) strategy y is pure strategy.

Best response to row players (mixed) strategy x is pure strategy.

proof: Best response computation via LP.

Given y, compute

$$\max x^T \cdot A \cdot y$$

$$\sum_{i=1}^{m} x_i = 1$$

$$x_i \geq 0 \quad i=1,\dots,m.$$

Basic solutions to LP are exactly the unit vectors

Similar argument for best response of column-player. □

Lemma: Given mixed strategies x and y, there are always pure strategies x' and y' with

$$x'^T \cdot A \cdot y \leq x^T \cdot A \cdot y$$

$$x^T B \cdot y' \leq x^T B \cdot y$$

proof: x and y are convex combinations of pure strategies (unit vectors) ■

<u>Theorem (Nash):</u> Every Two-person general sum Game (bi-matrix Game) has Nash equilibrium.

For the proof we need:

<u>Theorem:</u> (Brouwer's Fixpoint Theorem)

$S \subseteq \mathbb{R}^n$ compact convex and $f: S \to S$ continuous, then there exists at least one $x \in S$ with $f(x) = x$.

<u>Proof of Nash Theorem:</u>

Let x, y be mixed strategies.

$$r_i(x,y) = \max \{0, e_i^T A \cdot y - x^T A \cdot y\}$$

$$c_j(x,y) = \max \{0, x^T \cdot B \cdot e_j - x^T \cdot By\}$$

<u>Notice:</u> x, y Nash equilib. $\Leftrightarrow$ $r = c = 0$

New strategies: $$x_i' = \frac{x_i + r_i}{1 + \sum_k r_k}$$

$$y_j' = \frac{y_j + c_j}{1 + \sum_k c_k}$$

<u>Interpretation:</u> "Move" x and y ~~to~~ in improving direction, if not all r_i, c_j are zero

$T(x,y) = (x', y')$ is continuous.

$S = \{ (x,y) \in \mathbb{R}^{m+n} : x, y \text{ mixed strategies} \}$ is compact, convex.

Brouwer's Fixpoint Thm $\Rightarrow$ $\exists (x,y)$ with

$T(x,y) = (x,y)$

<u>Claim:</u> (x,y) with $T(x,y) = (x,y)$ is Nash equilibrium.

<u>proof of claim:</u>

Suppose (x,y) is not Nash equilibrium. Then:

$\exists i$ with $e_i^T \cdot A \cdot y > x^T \cdot A \cdot y$ or

$\exists j$ with $x^T \cdot B \cdot e_j > x^T \cdot B \cdot y$.

Assume first cond. holds. Then $r_i > 0$ and

~~Lemma~~ Thus $\sum_{k} r_i > 0$.

<u>Lemma</u> $\Rightarrow$ $\exists i'$ with $r_{i'} = 0$

$\Rightarrow$ $\exists i'$ with $x'_{i'} \neq x_{i'}$.

implying that (x,y) is not a fixpoint.

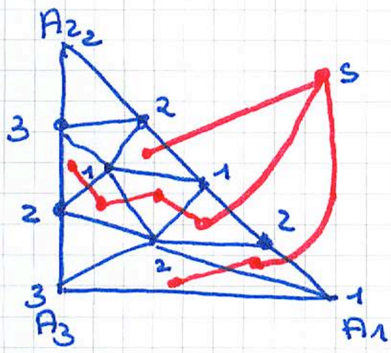
If second condition holds, then similar argument applies. $\Rightarrow$ claim

$\Rightarrow$ $\exists$ Nash equilibrium

Why does Brouwer's Fixpoint Thm. Hold?

We provide elementary Proof for the case, where $S \subseteq \mathbb{R}^2$ is a triangle. (unit triangle)

Sperner Lemmma:



- A triangle with ~~vertex labels~~ vertices A_1, A_2, A_3
- points in triangle have labels in $\{1,2,3\}$
- Each point on $\overline{A_iA_j}$ has labels in $\{i,j\}$
- Labels of A_i are i respectively.

Sperners Lemmme: Each Triangulation of points has rainbow triangle (labels 1, 2, and 3)

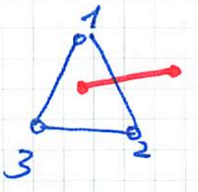
proof: Consider Graph induced by Triangulation. Draw only edges of Dual Graph that correspond to neighboring triangles with common 1,2 border

Recall that for graph $G=(V,E)$:

$$\sum_{v \in V} \deg(v) = 2E.$$

Degree of S is odd, since odd number of 1-2 swaps!

$\Rightarrow \exists$ other odd triangle



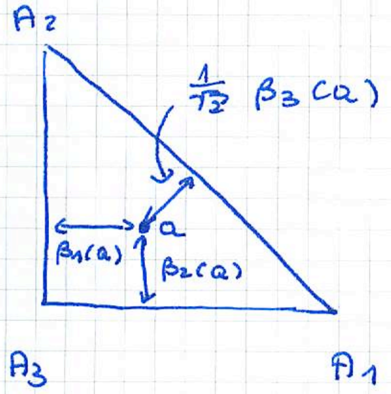
Rainbow Triangle!

$\square$

Brouwer's Fixpoint Thm for unit Triangle

$f: \Delta \to \Delta$ continuous, then f has a fixpoint

proof:



$a = (x, y)$

$\beta_1(a) = x$

$\beta_2(a) = y$

$\beta_3(a) = 1 - x - y$

We have:
1) $\beta_i(a) \geq 0 \quad \forall a \in \Delta$
2.) $\beta_1(a) + \beta_2(a) + \beta_3(a) = 1 \quad \forall a \in \Delta$

Label points in triangle with $\{1, 2, 3\}$

$$M_i := \{ a \in \Delta : \beta_i(a) \geq \beta_i(f(a)) \} \quad , M_1 \cup M_2 \cup M_3 = \Delta$$

We want:
$label(a) = i \implies a \in M_i \qquad (*)$

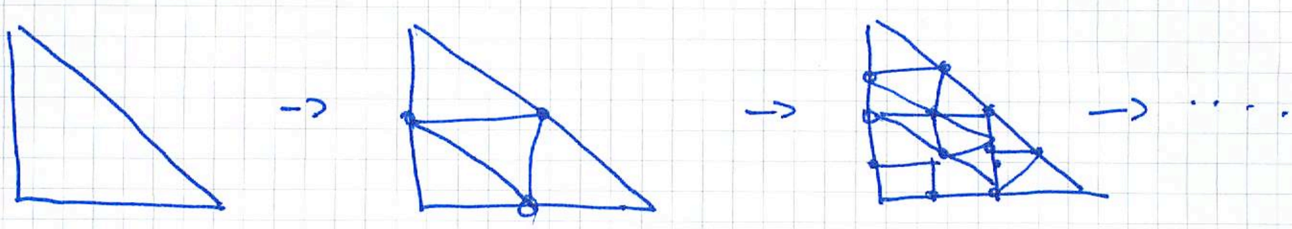
Notice $A_i \in M_i \implies label(A_i) = i$ is admissible.

Notice that line segment $\overline{A_i A_j} \in M_i \cup M_j \quad i, j \in \{1, 2, 3\}$

$\implies$ points on $\overline{A_i A_j}$ have label in $\{i, j\}$

=> Labeling (*) as in prerequisites of Sparner Lemma is possible.

Subdivide Triangle into smaller and smaller Triangles



Each subdivision contains rainbow triangle (Sparner)

Sequence of Rainbow triangles has converging subsequence (Bolzano-Weierstrass). Common 3-vertices of limit triangle is point $p \in \Delta$. (Δ is closed & bounded).

We have $\beta_i(p) \geq \beta_i(f(p))$ $i=1,2,3$

=> p is fixed point of f. $f(p)=p$

$\square$