Convexity

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Assignment Sheet 7

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Exercise 1
A polytope $P \subseteq \mathbb{R}^n$ is called *integral* if $P = \operatorname{conv}(P \cap \mathbb{Z}^n)$ holds. Show that a polytope is integral if and only if all of its vertices are integer points.

Exercise 2
Let $\Lambda \subseteq \mathbb{R}^n$ be a lattice, $\mathcal{V}$ its voronoi cell. For $p \in \Lambda$, denote the voronoi cell translated by p with $\mathcal{V}(p)$. Let $p \in \Lambda$ such that $\mathcal{V}(p) \cap (2\mathcal{V}) \neq \emptyset$.

1. Show that the line segment $[0, p]$ is contained in $2\mathcal{V} \cup \mathcal{V}(p)$.
2. Using the previous fact, argue that $\mathcal{V}(p) \subseteq (4\mathcal{V})$.
3. Show that at most 4^n of the voronoi cells $\mathcal{V}(p)$, $p \in \Lambda$, have a non-empty intersection with $(2\mathcal{V})^\circ$.

Exercise 3
Let $\Lambda \subseteq \mathbb{R}^n$ be a lattice, and $\mathcal{V}$ its voronoi cell. For $p \in \Lambda$, denote the voronoi cell translated by p with $\mathcal{V}(p)$. Assume that Λ is of such form that the following holds.

$(\star)$ Any point $p \in \Lambda$ for which $(2\mathcal{V})^\circ \cap \mathcal{V}(p)^\circ \neq \emptyset$ lies on the boundary of $2\mathcal{V}$.

Show that we have
$$|\{p \in \Lambda : \ (2\mathcal{V})^\circ \cap \mathcal{V}(p)^\circ \neq \emptyset\}| \leq 3^n.$$
Show that the property $(\star)$ is not true in general.

Exercise 4
Let $\Lambda \subseteq \mathbb{Z}^n$ be a lattice, $\mathcal{V}$ its voronoi cell, given in the form $\{x : \ Ax \leq b\}$, and $t \in \mathbb{R}^n$ be a target vector.

1. Using Exercise 2, show that if $t \in 2\mathcal{V}$ and the description $Ax \leq b$ of the voronoi cell is given, then a closest vector can be found in time $2^{O(n)}$ and state the algorithm.
2. State an algorithm with the following specifications. The input is a basis B of Λ and a target vector t, the output is a closest vector to t in Λ. Moreover, the algorithm has access to the description $Ax \leq b$ of the voronoi cell of Λ and has running time $\log(\|t\| + 1)2^{O(n)}$, assuming that each basic arithmetic operation can be performed in constant time.

Prove correctness for your algorithms.
[*Hint: What is the difference between the first and the second part? Which running time would you (roughly) get if you used the technique of the first part directly on the second part? Which techniques do you know to reduce the appearing factor? For example, how can you implement* $f(x, n) = x^n$ *only using* $O(\log(n + 1))$ *arithmetic operations, for n integral?*]

The deadline for submitting solutions is **Thursday, November 24, 2016**.