Strong Relaxations for Discrete Optimization Problems 29/04/16

Lecture : Extension complexity of stable set and matching polytopes

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1 The extension complexity of the stable set polytope

Let us recall the correlation polytope and the theorem we proved in a previous lecture on its extension complexity.

Definition 1 *We define the* correlation polytope $corr(n)$ *to be*

$$corr(n) = conv\{y^b : b \in \{0,1\}^n\} \subseteq \mathbb{R}^{n\times n},$$

where $y^b \in \mathbb{R}^{n\times n}$ *is the outer product* $y^b = bb^\top$.

Theorem 2 $xc(corr(n)) = 2^{\Omega(n)}$.

This result can be used as a starting point to prove similar lower bounds on the extension complexity of many interesting polytopes. Two useful observations to achieve this are the following.

Lemma 3 *Let* P *be a polyhedron and* F *a face of* P*. Then* $xc(F) \leq xc(P)$.

Proof From polyhedral theory we know that $F = \{x \in P : Ax = b\}$ for an appropriate system $Ax = b$. Hence, for any extended formulation $Cx + Dy \leq c$ of P, $Cx + Dy \leq c$, $Ax = b$ is an extended formulation for F of the same size. ■

Exercise 4 *Assuming that in previous lemma* P *and* F *are polytopes, prove the statement using properties of slack matrices.*

Lemma 5 *Let* $P \subseteq \mathbb{R}^n$, $Q \subseteq \mathbb{R}^{n+p}$ *be polyhedra such that* Q *is an extension of* P*. Then* $xc(P) \leq xc(Q)$.

Proof Clearly any extension of Q is also an extension of P, obtained by composing the linear maps. ■

Suppose now that we show the following reduction.

Lemma 6 *For each* $n \in \mathbb{N}$*, there exists a graph* G *with* $O(n^2)$ *nodes such that there exists a face* F *of* $STAB(G)$ *that is an extension of* $corr(n)$.

Then one can conclude using the previous lemmata that for any of those graph $G(V,E)$,

$$xc(STAB(G)) \geq xc(F) \geq xc(corr(n)) \geq 2^{\Omega(\sqrt{|V|})}, \text{ hence}$$

Theorem 7 $xc(STAB(G(V,E))) \geq 2^{\Omega(\sqrt{|V|})}$.

Proof *(of Lemma 6)*

Let $n \in \mathbb{N}$. Consider the graph $G(V,E)$ define as follows. For each index $i \in [n]$, create nodes ii and $\overline{ii}$. For each pair of indices $i,j \in [n]$ with $i < j$, create nodes $ii, \bar{\imath}j, \bar{\jmath}i, \bar{\imath}\bar{\jmath}$. Add edges as follows: for each $i \in [n]$, add edge $(ii, \overline{ii})$, for each pair of indices $i,j \in [n]$ with $i < j$, add all edges between nodes $ii, \bar{\imath}j, \bar{\jmath}i, \bar{\imath}\bar{\jmath}$, plus edges $(ii, \bar{\imath}j)$, $(ii, \bar{\imath}\bar{\jmath})$, $(\overline{ii}, i\bar{\jmath})$, $(\overline{ii}, ij)$, $(jj, i\bar{\jmath})$, $(jj, \bar{\imath}\bar{\jmath})$, $(\overline{jj}, \bar{\imath}j)$, $(\overline{jj}, ij)$.

We will now define a linear fuction π mapping a point $x \in \mathbb{R}^V$ to a point $y \in \mathbb{R}^{n\times n}$ that maps a face F of $STAB(G)$ to $corr(n)$. Hence, F is an extension of $corr(n)$, concluding the proof.

F is the face of $STAB(G)$ given by clique inequalities

$$x_{ii} + x_{\overline{ii}} \leq 1 \text{ for all } i \in [n] \text{ and } x_{ij} + x_{\overline{i}j} + x_{i\overline{j}} + x_{\overline{ij}} \leq 1 \text{ for all } i,j \in [n] \text{ with } i < j,$$

while the map π is defined as follows:

$$y_{ij} = y_{ji} = x_{ij} \text{ for } i \leq j.$$

Consider a vertex x of F: this is the characteristic vector of a stable set S of G satisfying at equality the clique inequalities above. In particular, for each $i < j$,

- either ii, jj, and $ij \in S$,
- or ii, $\overline{jj}$, and $i\overline{j} \in S$,
- or $\overline{ii}$, jj, and $\overline{i}j \in S$,
- or $\overline{ii}$, $\overline{jj}$, and $\overline{ij} \in S$.

In particular, $ij \in S$ iff $ii, jj \in S$. Let $b \in \{0,1\}^n : b_i = 1$ iff $ii \in S$. Then for $y = \pi(x)$ and $i \leq j$ we have $y_{ij} = x_{ij}$ iff $ii, ij \in S$, i.e., $y = bb^T$.

Conversely, consider point $y = bb^T$, with $b \in \{0,1\}^n$. Then the unique stable set of G that contains ii iff $b_i = 1$ belongs to F and is in the preimage of y, as required. ■

2 The extension complexity of the perfect matching polytope

Recall that, to prove Theorem 2, we used the concept of rectangle cover.

Definition 8 *Given a matrix $S \in \mathbb{R}^{m \times d}_{\geq 0}$, a* rectangle *$R = (X,Y)$ in S is a subset of rows X and a subset of columns Y such that all entries of the minor $S[X \times Y]$ are positive. If we define $supp(R)$ to be $X \times Y$, then in other words we want that $supp(R) \subseteq supp(S)$.*

Definition 9 *A family $\mathcal{R}$ of rectangles of S is called a* rectangle cover *if these rectangles together cover all positive entries of S, i.e.,*

$$\bigcup_{R \in \mathcal{R}} supp(R) = supp(S).$$

Theorem 10 *$rk_+(S) \geq \min\{|\mathcal{R}| : \mathcal{R}$ is a rectangle cover of $S\}$.*

We now show that rectangle covers are of no use for lower bounding the extension complexity of the perfect matching polytope. Recall that the following is a linear description of the perfect matching polytope of a graph $G(V,E)$.

$$PMATCH(G) = \{x \in \mathbb{R}^V : \begin{array}{lll} x(\delta(v)) & = 1 & \text{for all } v \in V, \\ x(\delta(U)) & \geq 1 & U \subseteq V, |U| \text{ odd and at least } 3, \\ x & \geq 0\}. & \end{array}$$

Note that $P(G)$ is always a face of $PMATCH(K_{|V|})$, with $K_{|V|}$ being the complete graph with $|V|$ nodes. Hence, because of Lemma 3, $xc(PMATCH(G)) \geq xc(PMATCH(K_{|V|}))$.

Lemma 11 *Let S be a slack matrix of the perfect matching polytope of K_n. Then there exists a rectangle cover of S with $O(n^4)$ rectangles.*

Proof It is enough to consider the slack matrix whose columns are indexed by matching of $K_n(V,E)$ and whose rows by inequalites $x(\delta(U)) \leq \frac{|U|-1}{2}$ for $U \subseteq V, |U|$ odd and at least 3. Note that the slack at (U,M) is non-zero iff there exists at least two edges of M that have exactly one endpoint in U. Each rectangle $R = P \times C$ is indexed by unordered pair of non-incident edges e_1, e_2, with $U \in P$ iff e_1 and e_2 have exactly one enpdpoint in U and $M \in P$ iff $e_1, e_2 \in M$. ■

Despite the previous lemma, and despite the fact that a maximum weight perfect matching in a graph can be computed in polynomial time, one can prove that the extension complexity of the perfect matching polytope is exponential.

Theorem 12 $xc(PMATCH(G(V,E))) = 2^{\Omega(|V|)}$.

The proof is quite involved, so here we will only give a glimpse of the techniques and refer the interested reader to the original paper.

The strategy is as follows: recall that the nonnegative rank of S is the minimum r such that S can be written as the sum of r nonnegative rank-1 matrices. In particular, the support of each of those matrices is contained in the support of S. Let $\mathcal{R}$ be the family of $0-1$ rank-1 matrices whose support is contained in the support of S. One first shows that there exists a matrix S that is a minor of the slack matrix of $PMATCH(K_n)$ such that $< W, S >$ is much bigger than $< W, R >$ for $R \in \mathcal{R}$ (here $< W, S >= \sum_{i,j} W_{i,j} S_{i,j}$ denotes the Frobenius product of W and S). We will then be done using the following *Hyperplane separation bound.*

Lemma 13 *Let $S \in \mathbb{R}^{m \times n}_{\geq 0}$, $W \in \mathbb{R}^{m \times n}$. Then*

$$rk_+(S) \geq \frac{< W, S >}{\alpha ||S||_\infty},$$

where $\alpha = \max\{< W, R >: R \in \mathcal{R}\}$.

Proof We first show that any rank-1 matrix $R \in [0,1]^{m \times n}$ is in the convex hull of matrices from $\mathcal{R}$. Write $R = uv^T$. Note that we can assume that all entries of u and v are between 0 and 1. Indeed, let $\Delta > 1$ be the maximum entry in, say, u. Then all entries of v are between 0 and Δ^{-1}, since $R \in [0,1]^{m \times n}$. We can then scale u by Δ^-1 and v by Δ. Let x, y be indipendent $0-1$ random vectors, distributed so that $P[x_i = 1] = u_i$ and $P[y_i = 1] = v_i$. We obtain

$$R = uv^T = \mathbb{E}[x]\mathbb{E}[y]^T = \sum_{\bar{u} \in \{0,1\}^m} P(x = \bar{u})\bar{u} \sum_{\bar{v} \in \{0,1\}^n} P(y = \bar{v})\bar{v}^T = \sum_{\bar{u} \in \{0,1\}^m, \bar{v} \in \{0,1\}^n} P(x = \bar{u}, y = \bar{v})\bar{u}\bar{v}^T,$$

hence probabilities $P(x = \bar{u}, y = \bar{v})$ give the nonnegative multipliers of the convex combination. Now let $S = \sum_{i=1}^r R_i$, where the R_i are rank-1 matrices. We conclude

$$< W, S >=< W, \sum_{i=1}^r R_i >= \sum_{i=1}^r < W, R_i >= \sum_{i=1}^r ||S_i||_\infty < W, \frac{R_i}{||R_i||_\infty} >= \sum_{i=1}^r ||R_i||_\infty \alpha \leq r||S||_\infty \alpha,$$

as required. ■

Let us remark that, unlike the rectangle covering bound, which only depends on the support of S, the hyperplane separation bound depends on the specific entries of S.

We now have to find a minor S of the slack matrix of $P(K_n)$ and a matrix W for which $< W, S >$ is much bigger than $< W, R >$ for each $R \in \mathcal{R}$. The columns of S will be indexed by the family $\mathcal{M}$ of all perfect matchings of K_n, while its rows will be indexed by a certain family $\mathcal{U}$ of sets $U \subseteq V$.

For $\ell = 1, \dots, n-1$, we let

$$Q_\ell = \{(U, M) \in \mathcal{U} \times \mathcal{M} : |M \cap \delta(U)| = \ell\}.$$

Note that entries in Q_1 have slack equal to 0. Hence, the support of any rectangle in $\mathcal{R}$ does not intersect Q_1. We now define the functional W over $\mathcal{U} \times \mathcal{M}$ as follows:

$$W_{U,M} = \begin{cases} \frac{1}{|Q_3|} & \text{if } (U, M) \in Q_3 \\ -\frac{1}{(k-1)|Q_k|} & \text{if } (U, M) \in Q_k \\ 0 & \text{otherwise.} \end{cases}$$

for an appropriate constant k. $< W, S >$ can be computed exactly

$$< W, S >= \frac{1}{|Q_3|}|Q_3|2 - \frac{1}{(k-1)|Q_k|}|Q_k|(k-1) = 1$$

Upper bounding $< W, R >$ for $R \in \mathcal{R}$ is much more complicated. Suppose one could prove the following (which, in fact, is hard to prove).

Theorem 14 *Let W, S be defined as above. Then for each $R \in \mathcal{R}$, one has $< W, S > \leq 2^{-\delta n}$ for some constant $\delta > 0$.*

Then using Lemma 13 and Theorem 14, and the properties of slack matrices, one concludes

$$\begin{aligned} xc(P(K_n)) \quad &\geq \quad rk_+(S) \quad \geq \quad \frac{< W, S >}{||S||_\infty \max\{< W, R >: R \in \mathcal{R}\}} \\ &\geq \quad \frac{1}{n2^{-\delta n}} \quad = \quad 2^{\Theta(n)}, \end{aligned}$$

as required.

What is the intuitive meaning of W? Note that, in the rectangle cover given by Lemma 11, entries of value k of the slack matrix with are covered $\binom{k+1}{2}$ times. Hence, in this rectangle cover entries with big value are over-covered. This is the reason why the rectangle cover cannot be transformed into a valid nonnegative factorization of the slack matrix. W is penalizing a rectangle R for each entry of Q_k it covers. Theorem 14 implies that all rectangles that covers many entries of Q_3 will have to cover many entries of Q_k as well. So all coverings with few rectangles of the slack matrix share with the rectangle covering of Lemma 11 the property of over-covering entries with big value (indeed, k here is fixed, but any constant k big enough would work for an appropriate δ).

3 Notes

Theorem 7 appeared in [2], Lemma 11 appeared in [1], while Theorem 12 appeared in [3]. In [3], Lemma 13 is attributed to Samuel Fiorini.

References

[1] S. Fiorini, V. Kaibel, K. Pashkovich, D. Theis. Combinatorial bounds on nonnegative rank and extended formulations. Discrete Mathematics 313(1), pages 67–83, 2013.

[2] S. Fiorini, S. Massar, S. Pokutta, H. R, Tiwary, and R. de Wolf. Linear vs. semidefinite extended formulations: exponential separation and strong lower bounds. In Proceedings of STOC 2012.

[3] T. Rothvoss. The matching polytope has exponential extension complexity. Proc. STOC 2014, pp. 263–272, 2014.