Strong Relaxations for Discrete Optimization Problems 15/04/16

Lecture : Yannakakis' theorem and the correlation polytope

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1 Recall extended formulations

Let us recall a couple of definitions on extended formulations.

Let $P \subseteq \mathbb{R}^n$ be a polyhedron. Then a system

$$\left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p : E^{=}x + F^{=}y = d^{=},\ E^{\leq}x + F^{\leq}y \leq d^{\leq}\right\} \tag{1}$$

is called an *extended formulation for P* iff

$$P = \{x \in \mathbb{R}^n : (\exists y \in \mathbb{R}^p)\ (x, y) \in Q\},$$

where Q is the polyhedron define by 1. In other words, P is the orthogonal projection of Q onto the subspace $\mathbb{R}^n \times \{(0, ..., 0)\}$. We also write $\pi_x(Q) = P$ [1]. Given P and Q as above, define the size of the extended formulation Q size(Q) to be the number of inequalities in its description (that is, the number of rows of the matrix $E^{\leq}$). We define the *extension complexity* of a polytope P to be

$$\text{xc}(P) = \min\{\text{size}(Q) : Q \text{ is an extended formulation for } P\}.$$

During last lecture we saw techniques for producing extended formulations, hence upper bounding $xc(P)$. In this lecture, we will show how to provide lower bound on the size of *any* extended formulation for a polytope.

2 A general approach

Even though extended formulations are not a new idea, for a long time only upper bounds (and some conditional lower bounds) were known. However, in recent years exponential lower bounds have been proved for a number of 0/1 polytopes. These developments are the product of a unified approach to extension complexity which we will present now. We will focus on polytopes. We also show a lower bound for one particular polytope: the correlation polytope.

2.1 The slack matrix and its nonnegative rank

The first step is to associate with P a matrix called the *slack matrix of P*, from which we will be able to extract information about the structure of the extended formulations of P.

Recall that any polytope can be written in two ways: as

$$P = \{x \in \mathbb{R}^n : Ax \leq b\}$$

for some matrix $A \in \mathbb{R}^{m \times n}$, with m being the number of constraints, or as

$$P = \text{conv}\{v_1, ..., v_d\}$$

with $v_1, ..., v_d \in \mathbb{R}^n$.

Definition 1 *Given both of these representations, the* slack matrix of P *is defined as a matrix* $S \in \mathbb{R}^{m \times d}$ *with*

$$S_{ij} = b_i - A_i v_j,$$

where A_i is the i-th row of A.

[1] With an abuse of notation, we will call extended formulation both the linear description of Q and Q itself. In the literature Q is usually called an *extension*

Since $b_i - A_i v_j$ is the slack of the i-th constraint at $v_j \in P$, we have $S_{ij} \geq 0$. Note that S is not unique for P, as it depends on the choice of $A, b, v_1, ..., v_d$.

Example 2 *Let $G = (V, E)$ be a complete graph on n vertices. Recall that the* matching polytope *of G can be expressed as*

$$M(n) = \left\{ x \in \mathbb{R}^{|E|} : x \geq 0, \quad x(\delta(v)) \leq 1 \textit{ for each } v \in V, \quad x(E(U)) \leq \frac{|U|-1}{2} \textit{ for each odd set } U \subseteq V \right\}$$

Consider the following slack matrix S of $M(n)$. Columns of S correspond to matchings M of G, while rows to nonnegativity inequalities (one per edge), to star inequalities (one per node), and inequalities $x(E(U)) \leq \frac{|U|-1}{2}$ for each odd set U. The entry of the slack matrix in column M and row corresponding to U is

$$S_{U,M} = \frac{|U|-1}{2} - |M \cap E(U)| \geq 0.$$

To see how S can be useful for us, we need another definition.

Definition 3 *Given any nonnegative matrix $S \in \mathbb{R}^{m \times d}_{\geq 0}$, we say that a pair of matrices (T, U) is a* rank-r nonnegative factorization *of S if:*

$$T \in \mathbb{R}^{m \times r}_{\geq 0}, \quad U \in \mathbb{R}^{r \times d}_{\geq 0}, \quad S = TU.$$

We define the nonnegative rank *of S as*

$$rk_+(S) = \min\{r : S \textit{ has a rank-}r \textit{ nonnegative factorization}\}.$$

Remark The name comes from the fact that if we skip "nonnegative" and "$_{\geq 0}$" everywhere in Definition 3, we will get an equivalent definition of the classical linear algebra rank of a matrix.

2.2 Yannakakis' Factorization Theorem

Now we can state the fundamental result which explains our interest in slack matrices.

Theorem 4 (Yannakakis' Factorization Theorem) *Let P be a polytope and S its slack matrix. Suppose* $\dim(P) \geq 1$. *Then*

$$xc(P) = rk_+(P).$$

We need a few lemmas for the proof:

Lemma 5 *If S' is a minor of S, then $rk_+(S') \leq rk_+(S)$.*

Proof Given a rank-r factorization $S = TU$ and a minor $S' = S[X \times Y]$, factorize $S' = T[X \times [r]][[r] \times Y]$. ■

Lemma 6 *For a matrix $S \in \mathbb{R}^{m \times d}_{\geq 0}$ we have $rk_+(S) \leq \min(m, d)$. Moreover, $rk_+(S)$ is at most the number of nonzero rows of S.*

Proof We have the following factorizations: $S = I_{m \times m} S = S I_{d \times d}$. For the second part, suppose that the nonzero rows of S are the first r rows; then write $S = JS'$, where $J \in \mathbb{R}^{m \times r}$ is the identity matrix $I_{r \times r}$ padded with $m - r$ zero rows and $S' = S[[r] \times [m]]$ is the nonzero part of S. ■

Lemma 7 *Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a polytope with $\dim(P) \geq 1$. If the inequality $cx \leq \delta$ is valid for P, then it can be written as a nonnegative combination of the constraints $Ax \leq b$, i.e.*

$$(\exists y \geq 0)\ yA = c,\ yb = \delta.$$

Proof

From Farkas' Lemma, we know that $cx \leq \delta'$ is a valid combination of constraints $Ax \leq b$, for some $\delta' \leq \delta$. In order to conclude the proof, it is enough to show that $0x \leq 1$ is also a nonnegative combination of $Ax \leq b$. Then we can take $cx \leq \delta'$ with multiplier 1, $0x \leq 1$ with multiplier $\delta - \delta' \geq 0$, and thus get $cx \leq \delta$.

Since $\dim(P) \geq 1$, at least one of the inequalities in $Ax \leq b$ is not always tight, otherwise P would be an affine subspace, and since it is bounded, it would be a point ($\dim(P) = 0$). That is, for some i,

$$\min\{A_i x : x \in P\} = b'_i < b_i,$$

where A_i is the i-th row of A. $b'_i \neq -\infty$ because P is bounded. Therefore the inequality $-A_i x \leq -b'_i$ is valid. Hence, $-A_i x \leq -b''_i$ can be obtained as a nonnegative combination of $Ax \leq b$, for some $b''_i \leq b'_i$. So the following inequalities can be so obtained: $-A_i x \leq -b''_i$, $A_i x \leq b_i$, and in consequence also their sum: $(A_i - A_i)x \leq b_i - b''_i$, which after scaling yields $0x \leq 1$.

■

Lemma 8 *Let*

$$P = \{x \in \mathbb{R}^n : Ax \leq b\} = conv\{v_1, ..., v_d\}$$

be a polytope with $\dim(P) \geq 1$ and S its slack matrix. Suppose $cx \leq \delta$ is a valid inequality for P. Then $rk_+(S) = rk_+(S')$, where S' is the matrix obtained by adjoining to S one extra row for the slack of the inequality $cx \leq \delta$.[2]

Proof Of course, by Lemma 5 we have $\mathrm{rk}_+(S) \leq \mathrm{rk}_+(S')$ and we need to prove the converse.

Use Lemma 7 to get a nonnegative combination

$$y \geq 0, \quad yA = c, \quad yb = \delta.$$

Denote $r = \mathrm{rk}_+(S)$ and take a nonnegative rank-r factorization

$$S = TU, \quad T \in \mathbb{R}^{m \times r}_{\geq 0}, \quad U \in \mathbb{R}^{r \times d}_{\geq 0}.$$

We claim that

$$S' = T'U, \quad T' \in \mathbb{R}^{(m+1) \times r}_{\geq 0}, \quad U \in \mathbb{R}^{r \times d}_{\geq 0},$$

where T' is T with the extra row yT adjoined, is a nonnegative rank-r factorization of S'. Indeed, if we denote the extra row of S' by β ($\beta^j = \delta - cv_j$), then we have $\beta = (yT)U$, because for each column j,

$$((yT)U)^j = (y(TU))^j = (yS)^j = yS^j = \sum_{i=1}^{m} y_i S_{ij} = \sum_{i=1}^{m} y_i (b_i - A_i v_j) = yb - yAv_j = \delta - cv_j = \beta^j.$$

■

Now we can prove Yannakakis' theorem.

Proof [of Theorem 4]

Denote the description of P from which S was created by:

$$P = \{x \in \mathbb{R}^n : Ax \leq b\} = \mathrm{conv}\{v_1, ..., v_d\}.$$

[2]In other words, S' is the slack matrix for the polytope P written as $P = \{x \in \mathbb{R}^n : Ax \leq b, cx \leq \delta\} = \mathrm{conv}\{v_1, ..., v_d\}$.

Direction $\mathbf{xc}(P) \leq \mathbf{rk}_+(S)$: given a rank-$r$ nonnegative factorization of S:

$$T \in \mathbb{R}_{\geq 0}^{m\times r}, \quad U \in \mathbb{R}_{\geq 0}^{r\times d}, \quad S = TU,$$

we can obtain an extended formulation of size r as follows:[3]

$$Q = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^r : Ax + Ty = b, \; y \geq 0\}.$$

Then:

- size$(Q) = r$ (note that there are many more equalities),
- we show that $\pi_x(Q) \subseteq P$: given $(x, y) \in Q$ we have $Ty \geq 0$ since $T \geq 0$ and $y \geq 0$, so $Ax = (Ax + Ty) - Ty = b - Ty \leq b$ and thus $x \in P$,
- we show that $\pi_x(Q) \supseteq P$: given $x \in P$, we need to produce a y such that $(x, y) \in Q$. Wlog assume that x is an extreme point of P – then we must have $x = v_i$ for some i. We take $y = U^i$ to be the i-th column of U. Then $y \geq 0$ and $Ax + Ty = Av_i + TU^i = Av_i + S^i = Av_i + b - Av_i = b$, so $(x, y) \in Q$.

Direction $\mathbf{xc}(P) \geq \mathbf{rk}_+(S)$: let

$$Q = \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p : E^= x + F^= y = d^=, \; E^\leq x + F^\leq y \leq d^\leq\right\}$$

be an extended formulation of P with size$(Q) = r$ (that is, the matrix $(E^\leq, F^\leq)$ has r rows). The objective is to show $\mathrm{rk}_+(S) \leq r$.

Observation: since $\pi_x(Q) = P$, all inequalities valid for P are also valid for Q. So our plan is the following: we will write down a slack matrix S' for Q, note that it has rank at most r, add the slacks of inequalities $Ax \leq b$ to it as extra rows (without increasing the rank), and then argue that S is a minor of the matrix so obtained, so S also has rank at most r.

Let then S' be the slack matrix of Q defined as follows. The first rows of S' contain the slack of inequalities $E^\leq x + F^\leq y$. The remaining rows contain the slack of constraints $E^= x + F^= y = d^=$, written as inequalities (two for each equality). Note that the latter are zero rows. As for the columns, we take any generating set of points of Q whose first d elements are in the preimages of points $v_1, ..., v_d$: denote them by $(v_1, y_1), ..., (v_d, y_d)$. From Lemma 6, $\mathrm{rk}_+(S') \leq r$.

Now add to S' rows corresponding to slacks of all inequalities $Ax \leq b$, obtaining a matrix S''. Because these inequalities are valid for Q, by Lemma 8, $\mathrm{rk}_+(S'') = \mathrm{rk}_+(S') \leq r$.

We are done if we prove that S is a minor of S'', since then by Lemma 5 we have $\mathrm{rk}_+(S) \leq \mathrm{rk}_+(S'') \leq r$. Indeed, S is the minor corresponding to the newly-added rows and to the first d columns. ■

2.3 Rectangle covers

Definition 9 *Given a matrix $S \in \mathbb{R}_{\geq 0}^{m\times d}$, a* rectangle $R = (X, Y)$ in S *is a subset of rows X and a subset of columns Y such that all entries of the minor $S[X \times Y]$ are positive. If we define supp(R) to be $X \times Y$, then in other words we want that supp$(R) \subseteq$ supp(S).*

Definition 10 *A family $\mathcal{R}$ of rectangles of S is called a* rectangle cover *if these rectangles together cover all positive entries of S, i.e.,*

$$\bigcup_{R\in\mathcal{R}} \mathit{supp}(R) = \mathit{supp}(S).$$

[3] On some level, this is the usual trick of adding nonnegative slack variables used to transform an LP of the form $Ax \leq b$ to one of the form $Ax = b, \; x \geq 0$. T having few columns means we do not have to add many slack variables.

We might think that a matrix which has no cover with a small number of rectangles has a complicated structure. Indeed:

Theorem 11 *$rk_+(S) \geq \min\{|\mathcal{R}| : \mathcal{R}$ is a rectangle cover of $S\}$.*

Proof Let $r = \mathrm{rk}_+(S)$ and write the nonnegative factorization:

$$S = TU = \sum_{l=1}^{r} T^l U_l$$

where T^l is the l-th column of T and U_l is the l-th row of U. Then

$$\mathrm{supp}(S) = \bigcup_{l=1}^{r} \mathrm{supp}(T^l U_l) = \bigcup_{l=1}^{r} \mathrm{supp}(T^l) \times \mathrm{supp}(U_l)$$

which yields a rectangle cover of size r. ■

Some remarks on this theorem are in order:

- One can ignore the exact values in the matrix – it only matters whether they are zero or not. This can often be helpful.
- The inequality is not tight – there are examples where $\mathrm{rk}_+(S)$ is exponential, but the minimum rectangle cover is only polynomial-sized.
- However, it will be useful for the correlation polytope, as we will see now.

3 The correlation polytope

Now we are well-equipped to prove an exponential lower bound.

Definition 12 *We define the* correlation polytope *$corr(n)$ to be*

$$corr(n) = conv\{y^b : b \in \{0,1\}^n\} \subseteq \mathbb{R}^{n \times n},$$

where $y^b \in \mathbb{R}^{n \times n}$ is the outer product

$$y^b = bb^\top.$$

Example 13

$$corr(2) = conv\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$$

Note that $\mathrm{xc}(\mathrm{corr}(n)) \leq 2^n$, for $\mathrm{corr}(n)$ is a convex hull of 2^n points (see Lemma 6). The rest of the lecture is devoted to the proof of the following theorem:

Theorem 14 *$xc(corr(n)) = 2^{\Omega(n)}$.*

We begin by relating the extension complexity of $\mathrm{corr}(n)$ to the nonnegative rank of a smartly defined matrix S.[4]

Definition 15 *Define the matrix $S \in \mathbb{R}_{\geq 0}^{2^n \times 2^n}$ as follows. Let its rows and columns be indexed by vectors $a, b \in \{0,1\}^n$, and write*

$$S_{ab} = \begin{cases} 0 & \text{if } |a \cap b| = 1, \\ 1 & \text{otherwise.} \end{cases}$$

[4]For the remainder of the notes, n will be suppressed in the notation.

See Figure 1 for an illustration of S.

Lemma 16 *$xc(corr(n)) \geq rk_+(S)$.*

Proof By Theorem 4 and Lemma 5, it is enough to show that the support of S is exactly the support of a minor of some slack matrix of $\text{corr}(n)$. For the column set of the minor, we will choose all the vertices of S (the matrices y^b). For the row set, we need to come up with a family of inequalities valid for $\text{corr}(n)$ parametrized by a such that the slack of the a-th inequality at y^b is 0 exactly when $|a \cap b| = 1$.

How to define the a-th inequality? Fix a, and consider first the following function of the variable $x = (x_1, ..., x_n) \in \{0,1\}^n$:

$$\pi_a : \{0,1\}^n \to \mathbb{Z}_{\geq 0}, \quad \pi_a(x) = (\langle a, x \rangle - 1)^2 \geq 0.$$

We linearize this function by expanding it into a multivariate polynomial in $x_1, ..., x_n$ and replacing all occurences of $x_i x_j$ with a variable y_{ij} and all occurences of x_i^2 (and x_i, which is equivalent since $x_i \in \{0,1\}$) with a variable y_{ii}. We thus obtain a linear functional $\rho_a : \mathbb{R}^{n \times n} \to \mathbb{R}$ with the property that $\rho_a(y^b) = \rho_a(bb^\top) = \pi_a(b)$ for any $b \in \{0,1\}^n$. Our choice of inequality is $\rho_a(y) \geq 0$. Note that it is valid for $\text{corr}(n)$, since for each vertex y^b we have $\rho_a(y^b) = \pi_a(b) \geq 0$, and that for any b we have that $\rho_a(y^b) = \pi_a(b) = 0$ iff $1 = \langle a, b \rangle = |a \cap b|$. ■

Now our task is reduced to showing that $\text{rk}_+(S)$ is exponentially large. We will do this by proving that S has no small *rectangle cover*.

3.1 The lower bound

Theorem 14 will follow from Theorem 11 if we can show that S cannot be covered by a small number of rectangles. We will actually show a somewhat stronger statement:

Theorem 17 *Let S' be the subset of positive entries of S defined as follows:*

$$(a, b) \in S' \Leftrightarrow |a \cap b| = 0.$$

Any collection $\mathcal{R}$ of rectangles in S which covers S' is of size at least $2^{\Omega(n)}$.

See Figure 1 for an illustration of S'.

Figure 1: The matrix S for $n = 3$. Entries in green are those belonging to S' (see Theorem 17).

$$\begin{array}{c} \\ 000 \\ 100 \\ 010 \\ 001 \\ 110 \\ 101 \\ 011 \\ 111 \end{array}
\begin{array}{c}
\begin{array}{cccccccc} 000 & 100 & 010 & 001 & 110 & 101 & 011 & 111 \end{array} \\
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1
\end{pmatrix}
\end{array}$$

Note that positive entries in S are those (a, b) with $|a \cap b| \neq 1$, so they have either $|a \cap b| = 0$ or $|a \cap b| \geq 2$. The set S' contains only the first ones. Recall that each rectangle of S can only contain positive entries, i.e. it contains no entry equal to 0.

To prove Theorem 17, let us first count how many entries there are to cover. The answer is simple: $|S'| = 3^n$, and this is because for each of the n elements we independently decide whether it is in a, in b, or in none of them.

So we will be done once we prove the following:

Theorem 18 *Any rectangle R in S covers at most 2^n entries of S'.*

This will imply that we need at least $3^n/2^n = (3/2)^n = 2^{\Omega(n)}$ rectangles to cover S'.

Proof [of Theorem 18]

For a rectangle R in S, we write $|R|_0 = |S' \cap R|$ (*magnitude* of R). We must prove $|R|_0 \leq 2^n$. (See Figure 2 for an example.)

Figure 2: An example rectangle $R = \{000, 101, 111\} \times \{000, 101, 111\}$, with $|R|_0 = 5 \leq 2^3$.

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} 000 & 100 & 010 & 001 & 110 & 101 & 011 & 111 \end{array} \\
\begin{array}{c} 000 \\ 100 \\ 010 \\ 001 \\ 110 \\ 101 \\ 011 \\ 111 \end{array} &
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1
\end{pmatrix}
\end{array}
$$

The proof is by induction on n. The case $n = 1$ is easy. Suppose we have the claim for $n - 1$.

Proof idea: we will cover $R \cap S'$ using two rectangles, and show that the magnitude of each of those rectangles is at most the magnitude of a rectangle which ignores the element n. By induction hypothesis, this is at most 2^{n-1}.

Fix a rectangle $R = P \times C$. Let

$$
\begin{aligned}
R_1 &= P_1 \times C_1, \\
P_1 &= \{a \in P : a \ni n\} \cup \{a \in P : a \cup \{n\} \notin P\}, \\
C_1 &= \{b \in C : b \not\ni n\}
\end{aligned}
$$

and let

$$\overline{R}_1 = \{(a \setminus \{n\}, b) : (a, b) \in R_1\}.$$

Note that $\overline{R}_1$ is a rectangle, for $\overline{R}_1 = \{a \setminus \{n\} : a \in P_1\} \times C_1$. Moreover, $\overline{R}_1 \subseteq \{0,1\}^{n-1} \times \{0,1\}^{n-1}$. So $|\overline{R}_1|_0 \leq 2^{n-1}$ by the induction hypothesis. Observe that $(a, b) \in S' \cap R_1$ if and only if $(a \setminus \{n\}, b) \in S' \cap \overline{R}_1$. In order to conclude $|R_1|_0 = |\overline{R}_1|_0 \leq 2^{n-1}$, it is then enough to show that for each $(a, b) \in \overline{R}_1$, exactly one of (a, b) and $(a \cup \{n\}, b)$ belongs to R_1. But this immediately follows by definition of P_1.

Define analogously:

$$
\begin{aligned}
R_2 &= P_2 \times C_2, \\
P_2 &= \{a \in P : a \not\ni n\}, \\
C_2 &= \{b \in C : b \ni n\} \cup \{b \in C : b \cup \{n\} \notin C\}
\end{aligned}
$$

and let

$$\overline{R}_2 = \{(a, b \setminus \{n\}) : (a, b) \in R_2\} \subseteq \{0,1\}^{n-1} \times \{0,1\}^{n-1}.$$

Then repeating the arguments above, we deduce $|R_2|_0 \leq 2^{n-1}$.

Claim: $R \cap S' \subseteq (R_1 \cup R_2) \cap S'$.

Once we have this, we conclude that $|R|_0 \leq |R_1|_0 + |R_2|_0 \leq 2^{n-1} + 2^{n-1} = 2^n$.

So let $(a, b) \in R \cap S'$. There are four cases:

- $a \ni n$, $b \not\ni n$: then $(a, b) \in R_1$,
- $a \not\ni n$, $b \ni n$: then $(a, b) \in R_2$,
- $a \ni n$, $b \ni n$: then $n \in a \cap b$ and so $|a \cap b| \neq 0$, a contradiction with $(a, b) \in S'$,
- $a \not\ni n$, $b \not\ni n$: if $a \cup \{n\} \notin P$ or $b \cup \{n\} \notin C$, then $(a, b) \in R_1$ or $(a, b) \in R_2$, respectively. So suppose that $a \cup \{n\} \in P$ and $b \cup \{n\} \in C$, which means that $(a \cup \{n\}, b \cup \{n\}) \in P \times C = R$. On the other hand, this pair cannot be covered by R, since the corresponding entry in S is zero. Indeed, since $(a, b) \in S'$, we have $a \cap b = \emptyset$ and thus $|(a \cup \{n\}) \cap (b \cup \{n\})| = |\{n\}| = 1$, so $(a \cup \{n\}, b \cup \{n\}) \notin R$. This contradiction concludes the proof.

■

4 Notes

Theorem 4 appeared in [3], while Theorem 14 appeared in [1]. Theorem 17 and the proof of Theorem 14 presented in here are essentially from [2], even if their proof of Theorem 14 is actually independent of Yannakakis' theorem.

These notes are essentially those that Jakub Tarnawski wrote for a lecture on similar topics I gave in another course. I thank Jakub for letting me use them. Any additional error is mine. YF

References

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