Recap

Linear Programming is an important tool in finance and economics

Goal: Develop an efficient method to solve linear programs

Recap

We assume that LPs are in standard form, where $A \in \mathbb{R}^{m \times n}$ has full row-rank:

$$\min c^T x$$
$$Ax = b$$
$$x \geqslant 0.$$

- $x^* \in \mathbb{R}^n$ is called basic solution, if $Ax^* = b$ and $\text{rank}(A_J) = |J|$, where $J = \{j \mid x^*(j) \neq 0\}$; Basic solution x^* is feasible basic solution, if $x^* \geqslant 0$
- Basis is set of indices $B \subseteq \{1, \ldots, n\}$ with $\text{rank}(A_B) = m$ and $|B| = m$
- $x^* \in \mathbb{R}^n$ with $A_B x_B^* = b$ and $x^*(j) = 0$ for all $j \notin B$ is basic solution associated to Basis B

Recap

- If LP has optimal solution, the LP has optimal basic solution
- We can enumerate all basic solutions and thus solve LPs (inefficient!)

Question

Do we have to enumerate all basic solutions?

Principle of Simplex Algorithm

Move from feasible basic solution to the next, thereby, if possible, improving the objective value, until optimal solution is found.

Moving a new index into the basis

- Consider Basis B with x^* feasible basic solution associated to B
- Choose index $j \notin B$
- Raise $x^*(j)$ from 0 to $\theta > 0$
  - The other non-basic variables $x(k), k \in \overline{B \cup \{j\}}$ shall remain unchanged
  - To maintain feasibility, the other basic variables might have to change
- Transition from x^* to $x^* + \theta \cdot d$

Requirements on d

- $d(j) = 1$
- $d(k) = 0,\ k \in \overline{B},\ k \neq j$
- $A(x^* + \theta \cdot d) = b$ for all $\theta > 0$; since $Ax^* = b$ and $\theta > 0$ this means $Ad = 0$.
- Alltogether $A_B \cdot d_B + a^j = 0$ and since B Basis (A_B invertible) one has $d_B = -A_B^{-1} a^j$

j-th basic direction

j-th basic direction

Let B be a basis and $j \notin B$. The vector d with $d(j) = 1$, $d(k) = 0$, $k \in \overline{B}$, $k \neq j$ and $d_B = -A_B^{-1} a^j$ is called j-th basic direction of B

At the transition from x^* to $x^* + \theta d$ the objective function changes by the amount

$$\theta \cdot c^T d = \theta(-c_B^T A_B^{-1} a^j + c(j)) = \theta \cdot (c(j) - c_B^T A_B^{-1} a^j)$$

The value $(c(j) - c_B^T A_B^{-1} a^j)$ represents the increase of the cost for $\theta = 1$.

Reduced costs

Let B be a basis. For $j \in \{1, \ldots, n\}$ the numbers

$$\overline{c}(j) = c(j) - c_B^T A_B^{-1} a^j$$

are called the reduced costs of the variables $x(j)$

Example

Consider the LP $\min\{c^T x \mid Ax = b, x \geqslant 0\}$ with

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 3 & 4 \end{pmatrix} \text{ and } b = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

- Choose $B = \{1, 2\}$
- $A_B = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 0 \end{smallmatrix}\right)$ is invertible and thus B is a basis
- Setting $x(3) = x(4) = 0$ and $x_B = A_B^{-1} b$ we obtain the feasible basic solution $x^* = (1, 1, 0, 0)$
- The third basic direction is obtained as follows.
  One has $d(3) = 1, d(4) = 0$ and

$$d_B = \begin{pmatrix} d(1) \\ d(2) \end{pmatrix} = -A_B^{-1} a^3 = -\begin{pmatrix} 0 & 1/2 \\ 1 & -1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 1/2 \end{pmatrix}$$

- The reduced cost of 3 is thus $\overline{c}(3) = -\frac{3}{2} c(1) + \frac{1}{2} c(2) + c(3)$.

A criterion for optimality

Lemma 1.3

Let x^* be a feasible basic solution associated to a basis B, If $\overline{c} \geqslant 0$, then x^* is an optimal solution of the LP.

Before we prove this Lemma, we come to a fundamental concept in linear programming. We will need it for example to prove the fundamental theorem of asset pricing.

The dual LP

Let

$$\min\{c^T x \mid Ax = b,\, x \geqslant 0\} \tag{2}$$

be an LP in standard form. The linear program

$$\max\{b^T y \mid A^T y \leqslant c\} \tag{3}$$

in inequality standard form is the dual program of the LP (2). The LP (2) is called the primal program.

Weak duality

Theorem (Weak duality)

Let x^* be a feasible solution of the primal and y^* be a feasible solution of the dual. One has

$$b^T y^* \leqslant c^T x^*. \tag{4}$$

Proof

Since $b = Ax^*$, one has

$$b^T y^* = x^{*T} A^T y^*.$$

Since furthermore $A^T y^* \leqslant c$ and $x^* \geqslant 0$, one has

$$x^{*T} A^T y^* \leqslant x^{*T} c = c^T x^*.$$

Together, we obtain the desired property:

$$b^T y^* \leqslant c^T x^*. \quad \square$$

Proof of Lemma 1.3

Recap: x^* is feasible basic solution defined by basis B and

$$\bar{c}(j) = c(j) - c_B^T A_B^{-1} a^j \geqslant 0 \text{ for all } j \in \{1, \dots, n\}. \tag{5}$$

(5) can be re-written as

$$c - A^T {A_B^{-1}}^T c_B \geqslant 0 \quad \Longleftrightarrow \quad A^T {A_B^{-1}}^T c_B \leqslant c. \tag{6}$$

$y^* = {A_B^{-1}}^T c_B$ is feasible solution of the dual $\max\{b^T y \mid A^T y \leqslant c\}$ LP of the primal $\min\{c^T x \mid Ax = b, x \geqslant 0\}$.
Objective value:

$$\begin{aligned} b^T y^* &= (Ax^*)^T y^* \\ &= (A_B x_B^*)^T y^* \\ &= (A_B x_B^*)^T {A_B^{-1}}^T c_B \\ &= {x_B^*}^T A_B^T {A_B^{-1}}^T c_B \\ &= c^T x^* \end{aligned}$$

Degeneracy

Question

Is the following assertion true:
If the reduced cost $\overline{c}(j) < 0$ for one $j \in \{1, \ldots, n\}$, then the feasible basic solution x^* is not optimal.

Answer is no!

- Let x^* be a feasible basic solution associated to basis B, $j \notin B$ and d the j-th basic direction.
- If $x^*(\ell) = 0$ for one $\ell \in B$, then $x^* + \theta \cdot d$ is for all $\theta > 0$ infeasible (If also $d(\ell) < 0$)

Such problematic basic feasible solutions are called degenerate

A basic solution x^* of a LP is called degenerate, if $|J| < m$, where $J = \{j \mid x^*(j) > 0\}$.

A limited converse of Lemma 1.3

Lemma 1.4

Let B be a basis and x^* a feasible basic solution associated to B. If x^* is non-degenerate and $\overline{c}(j) < 0$ for a $j \in \{1, \ldots, n\}$, then x^* is not optimal.

Proof

- Let $d \in \mathbb{R}^n$ be the j-th basic direction of B.
- One has $Ad = 0$, $d(j) = 1$ and $d(k) = 0$ for all $k \in \overline{B}$, $k \neq j$.
- $\overline{c}(j)$ denotes the increase of the objective function at the transition from x^* to $x^* + d$, Thus $\overline{c}(j) = c^T d$.
- Since $x_B^* > 0$ there exists a $\theta > 0$ with $x^* + \theta \cdot d \geqslant 0$ and therefore $x^* + \theta \cdot d$ is feasible solution.
- $c^T(x^* + \theta \cdot d) = c^T x^* + \theta \cdot c^T d < c^T x^*$

Optimal Basis

Optimal basis

Let $B \subseteq \{1, \dots, n\}$ be a basis of the LP $\min\{c^T x \mid Ax = b, x \geqslant 0\}$. B is optimal, if the following conditions hold:

- $A_B^{-1} b \geqslant 0$ basic solution is feasible
- $y^* = {A_B^{-1}}^T c_B$ is feasible dual solution or equivalently the reduced costs $\overline{c}$ of B are non-negative

Observation

An optimal basis yields an optimal basic feasible solution, whereas an optimal basic feasible solution can be associated to bases which are not optimal, since the reduced costs might be negative.

Iteration of simplex method (non-degenerate case)

x^* non-degenerate feasible basic solution with basis B; $\overline{c}$ the vector of the reduced costs

- If $\overline{c} \geqslant 0$, then x^* is optimal
- If $\overline{c}(j) < 0$ for a j then there exists $\theta > 0$ with $x^* + \theta \cdot d$ feasible, where d is j-th basic direction
- By increasing θ the variable $x(j)$ becomes positive and all other non-basic variables remain 0
- We say that j enters the basis
- Since the costs in direction d are strictly decreasing, one should advance as much as possible:

$$\theta^* = \max\{\theta \geqslant 0 \mid x^* + \theta \cdot d \text{ feasible}\}.$$

- The associated (negative) increase of the objective is: $\theta^* \cdot c^T d = \theta^* \cdot \overline{c}(j)$.
- Since $Ad = 0$, one has $A(x^* + \theta \cdot d) = Ax^* = b$
- The only danger to feasibility is caused by possibly violating the $x \geqslant 0$ constraint

We distinguish two cases:

a) If $d \geqslant 0$, then one has $x^* + \theta d \geqslant 0$ for all $\theta \geqslant 0$, the Vector $x^* + \theta d$ never becomes infeasible by increasing θ, one can choose $\theta = \infty$ The LP is unbounded

b) If $d(i) < 0$, for some $i \in \{1, \dots, n\}$, the constraint $x^*(i) + \theta d(i) \geqslant 0$ implies $\theta \leqslant -x^*(i)/d(i)$
Let $K \subseteq \{1, \dots, n\}$ be defined by $K = \{i \mid d(i) < 0\}$
Largest possible value for θ is given by formula

$$\theta^* = \min_{\{i \in K\}} -\frac{x^*(i)}{d(i)} \tag{7}$$

Since $d(i) \geqslant 0$, for $i \in \overline{B}$, one has $K \subseteq B$.
Observe that $\theta^* > 0$, since $x^*_B > 0$ (non-degenerate x^*)

Recap Example

Consider LP $\min\{c^T x \mid Ax = b, x \geqslant 0\}$ with

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 3 & 4 \end{pmatrix} \text{ and } b = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

- Choose $B = \{1, 2\}$. Then one has $A_B = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$, thus B is basis.
- Setting $x(3) = x(4) = 0$ and $x_B = A_B^{-1} b$ one obtains feasible basic solution $x^* = (1, 1, 0, 0)$
- 3-rd basic direction: One has $d(3) = 1, d(4) = 0$ and

$$d_B = \begin{pmatrix} d(1) \\ d(2) \end{pmatrix} = -A_B^{-1} a^3 = -\begin{pmatrix} 0 & 1/2 \\ 1 & -1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 1/2 \end{pmatrix}$$

- The reduced costs of this direction are $\overline{c}(3) = -\frac{3}{2} c(1) + \frac{1}{2} c(2) + c(3)$.

Example continued

- We assume that $c = (2, 0, 0, 0)^T$ thus $\overline{c}_3 = -3$
- Basic direction for $j = 3$ is $d = (-3/2, 1/2, 1, 0)$.
- Consider $y = x^* + \theta d$, for $\theta > 0$.
- Only component of y which is decreasing is y_1 (since $d(1) < 0$)
- Maximal value of θ is therefore $\theta^* = -x_1^* / d(1) = 2/3$
- New point $y^* = x^* + \frac{2}{3} d = (0, 4/3, 2/3, 0)$.
- The associated columns a^2, a^3 are linear independent and $\overline{B} = \{2, 3\}$ is new basis
- 1 has left the basis and 3 has entered the basis

Basis change

- If θ^* finite, we arrive at new feasible solution $y^* = x^* + \theta^* d$
- Since $x(j)^* = 0$ and $d(j) = 1$, one has $y(j)^* = \theta^* > 0$
- Let ℓ be an index which attains minimum in (7)
- We then have $d(\ell) < 0$ and $y^*(\ell) = x(\ell)^* + \theta^* d(\ell) = 0$
- Basic variable $x(\ell)$ becomes 0 while non-basic variable $x(j)$ is now positive
- This motivates to replace in B the index ℓ with j

$$B' = (B \setminus \{\ell\}) \cup \{j\}. \tag{8}$$

B' is basis

Theorem 1.5

a) $\text{rank}(A_{B'}) = m$

b) $y^* = x^* + \theta^* d$ *is basic solution associated to* B'

Proof

a) We need to show that columns of $A_{B'}$ are linearly independent
Consider $M = \{i \mid d(i) \neq 0, i \neq j\} \subseteq B$ and observe that $\ell \in M$
One has (since $Ax^* = b$)

$$a^j = -\sum_{i \in M} d(i) a^i \tag{9}$$

Assume: Vectors a^i, $i \in B'$ are linearly dependent

Proof cont.

Since columns of A_B and in particular those of $A_{B\setminus\{\ell\}}$ linearly independent, a^j must be in the span of the vectors a^i, $i \in B\setminus\{\ell\}$ Consequently, there are real numbers $\nu(i)$, $i \in B\setminus\{\ell\}$ with

$$a^j = \sum_{i\in B\setminus\{\ell\}} \nu(i) a^i. \tag{10}$$

Via subtracting equation (9) from the equation (10) we obtain

$$\sum_{i\in(B\setminus\{\ell\})\setminus M} \nu(i) a^i + \sum_{i\in M\setminus\{\ell\}} (\nu(i) + d(i))\, a^i + d(\ell) a^\ell = 0 \tag{11}$$

which shows that the columns of A_B are linearly dependent (contradiction!)

b) Is clear, since y^* is basic feasible solution associated to B'.
$(y^*(i) \neq 0 \implies i \in B')$ □

One iteration of Simplex Algorithm

1. Start with basic feasible solution x^* associated to basis B
2. Compute reduced costs $\overline{c}(j) = c(j) - c_B^T A_B^{-1} a^j$, for all $j \in \overline{B}$
3. If $\overline{c} \geqslant 0$, then is x^* optimal and Simplex terminates
4. Otherwise choose $j \in \overline{B}$ with $\overline{c}(j) < 0$
5. Compute $d_B = -A_B^{-1} a^j$
6. If $d_B \geqslant 0$, then $\theta^* = \infty$ and optimal value of LP is $-\infty$ and Simplex terminates
7. If $K = \{j \in B \mid d(j) < 0\} \neq \emptyset$ set

$$\theta^* = \min_{i \in K} -\frac{x^*(i)}{d(i)}$$

8. Choose ℓ such that $\theta^* = -\frac{x(\ell)}{d(\ell)}$ and replace ℓ by j in B
9. New feasible basic solution is $y^* = x^* + \theta \cdot d$

Simplex Algorithm terminates in non-degenerate case

Theorem 1.6

If $\min\{c^T x \mid Ax = b, x \geqslant 0\}$ *is LP with nonempty feasible solution set such that no feasible basic solution is degenerate, then the simplex algorithm terminates after finitely many steps and outputs one of the following:*

a) *An optimal basis with the associated optimal basic feasible solution* x^* *and a feasible dual solution* y^* *such that the objective values* $c^T x^*$ $b^T y^*$ *are the same*

b) *A direction* d *with* $Ad = 0, d \geqslant 0$ *und* $c^T d < 0$ *and the optimal value is* $-\infty$ *(the LP is unbounded).*

Strong duality in non degenerate case

Theorem 1.7

Let $\min\{c^T x \mid Ax = b, x \geqslant 0\}$ *be a linear program with nonempty feasible set such that no feasible basic solution is non degenerate. If the LP is bounded, then there exists an optimal solution* x^* *and an optimal solution* y^* *of the dual LP* $\max\{b^T y \mid A^T y \leqslant b\}$ *and the objective values are the same*

$$c^T x^* = b^T y^* .$$

Perturbation

What can we do if LP is degenerate

Suppose LP has degenerate basic feasible solutions

$$
\begin{aligned}
\min c^T x \\
Ax = b \\
x \geqslant 0
\end{aligned}
\tag{12}
$$

Idea: Modify LP by perturbing b slightly. Replacing b with b' yields new LP, LP' Here, the following shall be ensured:

i) LP' has feasible solution
ii) If B is non-feasible basis of LP, then B is non-feasible basis of LP'
iii) LP' does not have degenerate basic solutions

Perturbation

Let B be feasible basis of LP with $x_B^* = A_B^{-1} b \geqslant 0$ (feasibility)

Set $b' = b + A_B \begin{pmatrix} \varepsilon \\ \varepsilon^2 \\ \vdots \\ \varepsilon^m \end{pmatrix}$ where $\varepsilon > 0$

Property i) holds, since $A_B^{-1} \left(b + A_B \begin{pmatrix} \varepsilon \\ \varepsilon^2 \\ \vdots \\ \varepsilon^m \end{pmatrix} \right) = x_B^* + \begin{pmatrix} \varepsilon \\ \varepsilon^2 \\ \vdots \\ \varepsilon^m \end{pmatrix} \geqslant 0$

Property ii)

$\widetilde{B}$ infeasible basis, then $(A_{\widetilde{B}}^{-1} b)(i) < 0$ for one index i
If $\varepsilon > 0$ small enough, then

$$A_{\widetilde{B}}^{-1} \left(b + A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix} \right)(i) < 0$$

which implies $\widetilde{B}$ also infeasible for LP'

Property iii)

Let $\widetilde{B}$ basis of LP'

Consider associated basic solution

$$x^*_{\widetilde{B}} = A^{-1}_{\widetilde{B}} b + A^{-1}_{\widetilde{B}} A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix} \tag{13}$$

Each component is polynomial in indetermined ε

Each polynomial is $\neq 0$ since $\text{rank}(A^{-1}_{\widetilde{B}} A_B) = m$

Polynomial of degree m has at most m roots

If $\varepsilon > 0$ small enough, then no component is zero

Simplex on perturbed problem

- Simplex terminates on perturbed problem, since LP' is not degenerate
- There are two cases
- Simplex computes optimal basis of LP' and thus optimal basis of LP
- Simplex finds j-th basic solution d with $\overline{c}(j) = c^T d < 0$ and $d \geqslant 0$ certifying that LP' and LP are unbounded

Strong duality

Theorem 1.8

Let $\min\{c^T x \mid Ax = b, x \geqslant 0\}$ *be an LP in standard form. If LP is feasible and bounded, then also the dual LP* $\max\{b^T y \mid A^T y \leqslant c\}$ *is feasible and bounded and there exist optimal solutions* x^* *and* y^* *of the primal resp. dual with* $c^T x^* = b^T y^*$

Proof

Simplex terminates if started on LP' with optimal basis B of LP' (and of LP) if LP is bounded. The reduced cost of B yield optimal solution $y^* = A_B^{-1} c_B$ of the dual LP (of LP).