## Solutions 12.

**12-1:** We try to obtain $a_n$ as the linear combination of two geometric sequences. Thus, we write $a_n = A\lambda^n + B\mu^n$, and our goal is to find the real numbers $\lambda, \mu, A, B$. Then the recurrence relation yields $A\lambda^n(\lambda^2 - 5\lambda + 6) + B\mu^n(\mu^2 - 5\mu + 6) = 0$ for all non-negative integers $n$. Thus, assuming that neither $\lambda$, nor $\mu$ is zero, we have that $\lambda$ and $\mu$ are the roots of the quadratic polynomial $x^2 - 5x + 6 = (x-3)(x-2)$. So, we may take $\lambda = 3$ and $\mu = 2$. Thus, we obtain $a_n = A3^n + B2^n$. By $a_0 = a_1 = 1$, we obtain $a_n = -3^n + 2 \cdot 2^n$.

Note that to find the generating function of the sequence, we would use the recurrence relation to obtain $F(x) - 5xF(x) + 6x^2F(x)$ has 0 coefficients for all $x^n$ where $n \geq 2$. Using the initial values, we get 1 and -4 for the coefficients of 1 and $x$. So, $F(x) - 5xF(x) + 6x^2F(x) = 1 - 4x$. Thus, $F(x) = \frac{1-4x}{1-5x-6x^2}$.

**12-2:**

(a) We can prove the statement by induction on $n$. The base case $n = 1$ is trivial. Assume that the statement holds for all $1 \leq k \leq n+1$, we now show that it also holds for $n+2$. Indeed, we have $F_{n+2} = F_{n+1} + F_n$. By the induction hypothesis, $F_{n+1} - 1 = F_1 + \cdots + F_{n-1}$, therefore, $F_{n+2} - 1 = F_1 + \cdots + F_n$.

(b) We can prove $F_{n+m} = F_{n-1}F_m + F_nF_{m+1}$ by induction on $m$. The base cases $m = 1$ $m = 2$ are trivial. Assume that the statement holds up to $m-1$, we now show that it also holds for $m$. Indeed,

$$\begin{aligned} F_{n+m} &= F_{n+m-1} + F_{n+m-2} \\ &= F_{n-1}F_{m-1} + F_nF_m + F_{n-1}F_{m-2} + F_nF_{m-1} \\ &= F_{n-1}F_m + F_nF_{m+1}. \end{aligned}$$

(c) By using part (b), this part can be solved by induction on $k$.

**12-3:**

(a) We prove this for any prime $p$. Again, let $v_i \in \mathbb{F}_p^n$ be the characteristic vector of $A_i$ for every $i$. We will prove that the $v_i$ are linearly independent. Suppose $\sum_{i=1}^n \alpha_i v_i = 0$ for some $\alpha_i \in \mathbb{F}_p$, and take the inner product of this equation with $v_j$:

$$0 = \langle \sum_{i=1}^n \alpha_i v_i, v_j \rangle = \sum_{i=1}^n \alpha_i \langle v_i, v_j \rangle.$$

Here $\langle v_i, v_j \rangle = |A_i \cap B_j| \mod p = 0$ if $i \neq j$, and $\langle v_i, v_j \rangle = |A_j| \neq 0$ if $i = j$ by the assumption. Hence the above equation gives $0 = \alpha_j |A_j|$. As $\mathbb{F}_p$ is a field, this is only possible if $\alpha_j = 0$. Once again, we get that only the trivial linear combination gives zero, and the $v_i$ are independent.

Alternatively, one can prove independence over $\mathbb{Q}$ by choosing the $\alpha_i$ to be integers not all divisible by $p$ (this is possible – if not all of them are zero – by multiplying all of them by some number)

(b) Let us assign the sets into two families $X$ and $Y$: we put $A_i$ in $X$ if $|A_i|$ is not divisible by 2 and in $Y$ if $|A_i|$ is not divisible by 3. We might put some sets into

both of $X$ and $Y$, but the important thing is that every $A_i$ sits in some of $X$ and $Y$. (Otherwise $|A_i|$ would be divisible by 2 and 3, i.e., by 6.)
Now for any sets $U, V \in X$, we know that $|U \cap U|$ is not divisible by 2, but $|U \cap V|$ is divisible by 2 (even by 6). Applying part (a) with $p = 2$, we see that $X$ contains no more than $n$ sets. We can similarly apply part (a) with $p = 3$ to $Y$ and see that at most $n$ sets are contained in $Y$. As every $A_i$ is in at least one of $X$ and $Y$, we get that there are at most $2n$ sets in total.

**12-4:** Suppose the $v_i$ are linearly independent over $\mathbb{F}_p$ and extend them to a basis $v_1, \dots, v_d$. Let $A$ be a matrix whose $i$'th column is $v_i$. As the $v_i$ are linearly independent, this matrix has a nonzero determinant. But the determinant of $A$ is the same over $\mathbb{R}$ (modulo $p$), so the matrix is non-singular over $\mathbb{R}$: its columns form a basis. In particular, they are independent over $\mathbb{R}$.

*Alternative solution:* We saw in class that it is enough to show that the vectors are independent over $\mathbb{Q}$. Suppose $\sum_{i=1}^m \alpha_i v_i = 0$ for some $\alpha_i \in \mathbb{Q}$ not all zero. Then we can multiply this equation by the denominators of the $\alpha_i$ to get $\sum_{i=1}^m a_i v_i = 0$ for some integers $a_i$, not all 0. Let $p^r$ be the highest power of $p$ divisible by each $a_i$. Then $b_i = a_i/p^r$ is an integer for every $i$, not all the $b_i$ are divisible by $p$, and $\sum_{i=1}^m b_i v_i = 0$. Looking at this equation modulo $p$, we get a nontrivial linear combination of the $v_i$ over $\mathbb{F}_p$ that vanishes. This contradiction shows that the $v_i$ are independent over $\mathbb{Q}$.

**12-5:** Let $b_n = \log_2(a_n)$. Then we obtain

$$b_{n+2} = \frac{b_{n+1} + b_n}{2}.$$

Using the same method as in the exercise 1, we obtain $b_n = -\frac{4}{3}\left(\frac{-1}{2}\right)^n + \frac{7}{3}$. Thus, $a_n = 4\sqrt[3]{2} \cdot 2^{-\frac{4}{3}\left(\frac{-1}{2}\right)^n}$. Finally, $\lim_{n\to\infty} a_n = 4\sqrt[3]{2}$.