

## Solutions 8.

**8-1:** Consider the new set family obtained by taking the complement of each set, that is,  $\mathcal{G} := \{[n] \setminus F : F \in \mathcal{F}\}$ . Then  $\mathcal{G}$  is an intersecting family, and we can apply the Erdős-Ko-Rado theorem.

**8-2:** One possibility is to list all the possible 3-element sets that are not in the Fano plane and verify that each of them is disjoint from at least one set in the Fano plane.

Another solution is the following. Assume the family is not maximal, that is, there is a 3-element set  $X$ , which intersects every member of the family. Since  $X$  intersects every member of the family, then it will intersect the sets  $\{a, b, c\}$ ,  $\{c, e, f\}$  and  $\{a, g, f\}$ . Note that  $X$  cannot be the set  $\{a, c, f\}$ , since it would not intersect the set  $\{b, g, e\}$ , which is already in the family. Also, note that at least one of  $a, c$ , or  $f$  has to be in  $X$ . That means, that one of  $a, b$ , or  $c$  is not contained in  $X$ , and one of them is. We may assume that  $a \notin X$  and  $c \in X$ .

$X$  must intersect the set  $\{c, e, f\}$  (which is a member of the Fano plane), and therefore either  $e$  or  $f$  belongs to  $X$ . Note that  $X$  cannot contain both  $e$  and  $f$ , since, in this case,  $X = \{a, e, f\}$ , which is disjoint from  $\{c, d, g\}$ .

Assume  $e$  is in  $X$  and  $f$  is not. Since  $\{b, d, f\}$  is a set of the Fano plane, and  $f$  is not contained in  $X$ , then either  $b \in X$  or  $d \in X$ . On the other hand, using the above, if  $d$  belongs to  $X$ , then  $X = \{a, e, d\}$ , which was already a member of the Fano plane. Therefore  $b \in X$ , so  $X = \{a, e, b\}$ . However, in this case  $\{a, b, e\}$  cannot intersect  $\{c, d, g\}$ , so we reach a contradiction.

Assume now  $f$  is in  $X$  and  $e$  is not, and thus  $X$  must contain  $a, f$  and one more other element. Since the set  $\{c, d, g\}$  is in the Fano plane, and, since we have assumed that  $c \notin X$ , we obtain that  $X$  is either  $\{a, f, d\}$  or  $\{a, f, g\}$ . In both cases we find sets from the Fano plane that do not intersect  $X$ .

Since we have reached a contradiction, the Fano plane must be a maximal intersecting family.

Although the family of sets is maximal, its cardinality is only 7, which is less than the bound in the Erdős-Ko-Rado theorem - which is  $\binom{7-1}{3-1} = 15$ .

**8-3:** Note that  $n = 2k$ . For any set  $F$  of size  $k$ , consider its complement  $\bar{F}$ , and call the two a *pair*:  $F, \bar{F}$ . For each pair, we can pick one of the members of the pair, and obtain an intersecting family. There are  $\binom{n}{k}/2$  pairs.

We will first select  $n$  sets  $A_1, \dots, A_n$  whose intersection is empty. Assume we selected  $A_1, \dots, A_{j-1}$  with  $j \leq n$ . There are  $\binom{n}{k}/2 - (j-1) > 0$  pairs out of which we have not selected a set. Pick any of those pairs and let  $A_j$  be the member of that pair that does not contain the element  $j$ . Clearly, at the end we obtain the  $n$  sets as needed. We can now select one element from every pair we did not use (in an arbitrary manner), and obtain a family with  $\frac{1}{2}\binom{n}{k} = \frac{1}{2}\binom{2k}{k} = \binom{2k-1}{k-1}$  sets which do not have a common element, as needed.

**8-4:** We need to verify that Hall's condition holds, that is, that for any subset  $X$  of  $X_k$  the number of its neighbors in  $X_{k+1}$  is at least  $|X|$ . A member of  $X$  has  $n - k$  neighbors in  $X_{k+1}$ . Each member of  $X_{k+1}$  is a neighbor of  $k + 1$  members of  $X_k$ . Thus, the total number of neighbors of  $X$  is at least  $|X| \frac{n-k}{k+1} \geq |X|$ , as needed.

To deduce Sperner's theorem, we do the following. Let  $\mathcal{F}$  be an antichain in the power set of  $[n]$ , that is, a Sperner family. Consider the *levels* of  $\mathcal{F}$ , that is the subfamilies of  $\mathcal{F}$  of given sizes. Let  $k$  be the lowest level, that is, assume that we have size  $k$  sets in  $\mathcal{F}$ , but no smaller sets. Assume that  $k < n/2$ . Then take a matching as described above between  $X_k$  and  $X_{k+1}$ , and replace each size  $k$  member of  $\mathcal{F}$  by its pair in the matching. It is easy to see that the new set family will still be an antichain.

With this algorithm, we may achieve that the lowest level of  $\mathcal{F}$  is sets of size  $\lfloor n/2 \rfloor$ . With the same procedure, we can achieve that the highest level is also  $\lfloor n/2 \rfloor$ . On the other hand, there are only  $\binom{n}{\lfloor n/2 \rfloor}$  members in total on that level.

**8-5:** Suppose for a contradiction that for each element  $i \in [n]$ , there is a set  $F_i \in \mathcal{F}$  with  $i \notin F_i$ . Take  $F_1 = \{i_1, \dots, i_k\}$ . The following  $k + 1$  members  $F_1, F_{i_1}, F_{i_2}, \dots, F_{i_k}$  of  $\mathcal{F}$  have empty intersection, a contradiction.