Solutions 7.

7-1: If G is bipartite, then it clearly does not contain an odd cycle. For the opposite direction, we may assume that G is connected, otherwise, we can find a bipartition of each connected component. Fix an arbitrary vertex u, and color it red. Then, for any vertex v, take a path from u to v, and color v red or blue according to whether the path is of even or of odd length. It is easy to see that, since G contains no odd cycle, either all paths from u to v are of odd length, or all are of even length. Clearly, every edge of G has one vertex in the red set and one in the blue.

7-2: We need to check that Hall's condition holds for A. Take any subset $X \subset A$, let $B(X)$ be its neighborhood in B and look at the number of edges connecting X and $B(X)$. On the one hand, it is exactly $|X|k$ because every vertex in X has degree k. On the other hand, it is at most $|B(X)|l$ because every vertex in $B(X)$ has degree l (so at most l of them go to X). Hence $|X|k \leq |B(X)|l \leq |B(X)|k$ (using $l \leq k$), implying $|X| \leq |B(X)|$ for every X.

7-3: One can prove this by following the same proof we used to prove Hall's theorem. Here we show a tricky reduction to Hall's theorem instead.

Add a new vertex v to B and connect it to every vertex in A. Let us call this new graph G'. We claim that G' will satisfy Hall's condition. Indeed, take any subset Y of A and define $N_{G'}(Y)$ to be its neighborhood in G'. Then $N_{G'}(Y) = N(Y) \cup \{v\}$, so $|N_{G'}(Y)| = |N(Y)| + 1 \geq |Y|$. So there is a matching M in G' that contains every vertex of A. But at most one edge $e \in M$ touches the new vertex v, so $M - e$ is a matching in G that contains all but at most one vertex of A.

7-4: Clearly, the number of colors needed for an edge coloring is at least $\Delta(G)$.

Assume that using $\Delta(G)$ colors, we colored the maximum number of edges. That is, we have $\Delta(G)$ matchings of G, $M_1, \ldots, M_{\Delta(G)}$ containing the maximum number of edges. Suppose for a contradiction that there is an edge $e = \{u, v\}$ of G that does not belong to any M_i. Clearly, there is an M_i, say M_1, such that no edge of M_1 is incident with u, since u has degree at most $\Delta(G)$. Similarly, there is an M_j, say M_2, such that no edge of M_2 is incident with v.

Let C be the following subgraph of G: $C = M_1 \cup M_2$. In C, the degree of every vertex is at most two, and in particular, the degree of u and of v is 1. Thus, C is the union of disjoint paths, and cycles. Let P denote the path containing u. Clearly, P has edges alternating from M_1 and from M_2. More precisely, walking along P starting at u, we first have an edge in M_2, then in M_1, etc.

Now, P does not contain v (convince yourself of it!).

So, P ends in some vertex other than v. Now, swap the colors of edges of P. More precisely, let $M_1' := (M_1 \setminus (P \cap M_1)) \cup (P \cap M_2)$, and $M_2' := (M_2 \setminus (P \cap M_2)) \cup (P \cap M_1)$. This way, M_1' and M_2' are still matchings, and in total contain the same number of edges as M_1 and M_2. However, now we can add e to M_2', and still have a matching. Now, $M_1', M_2' \cup \{e\}, M_3, M_4, \ldots, M_{\Delta(G)}$ is a coloring of the edges containing one more edge than $M_1, \ldots, M_{\Delta(G)}$, a contradiction.

7-5: A perfect matching in G corresponds to a permutation π on $[n]$: for each i, $\pi(i) = j$

if the edge $\{i, j\}$ is in the matching. Thus, to count the number of perfect matchings, we need to count the number of those permutations that correspond to edges of G. A given permutation π corresponds to n edges of G if, and only if $a_{1\pi(1)}a_{2\pi(2)} \cdot \ldots \cdot a_{n\pi(n)} = 1$.

7-6: (a) Fix any vertex v of Q_n . All its neighbors differ from v in exactly one position. There are n positions possible to differ at. Hence, every vertex has degree n. Since, the number of edges e satisfies $2e = \sum_v deg(v) = n2^n$, we have $e = n2^{n-1}$.

(b) $Q_1 = K_2$ is connected. We use induction on n. Assume that Q_{n-1} is connected for some $n > 1$, and let's look at Q_n. We split its vertices to two sets: V_0 contains all the vertices of Q_n ending with 0 and V_1 contains all the vertices of Q_n ending with 1. Clearly, V_0 and V_1 are disjoint and every vertex of Q_n is in one of them.

Observe that both V_0 and V_1 are isomorphic to Q_{n-1}, thus, they are connected by the induction hypothesis.

It is sufficient to find one edge connecting V_0 and V_1 to see that Q_n is connected. And, in fact, there are many such edges.

7-7: Let V_o (resp. V_e) be the set of vertices of Q_n that have an odd (resp. even) number of ones. It is easy to see that these two sets give a desired partition of the set of vertices of V.