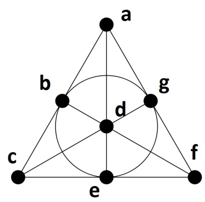
Problem Set 8.

8-1: Let $k \geq n/2$ and let $\mathcal{F}$ be a family of k-element subsets of $\{1, 2, \ldots, n\}$ such that $A \cup B \neq \{1, 2, \ldots, n\}$, for all $A, B \in \mathcal{F}$. Prove that $|\mathcal{F}| \leq \binom{n-1}{k}$.

8-2: The Fano plane is a set family on the base set $\{a, b, \ldots, g\}$ consisting of the following 7 triples: $\{a, b, c\}$, $\{a, d, e\}$, $\{a, f, g\}$, $\{b, d, f\}$, $\{b, e, g\}$, $\{c, d, g\}$, $\{c, e, f\}$. In the figure, the points represent the elements of the base set, while the lines and the circle represent members of the set family.



Prove that the Fano plane is a maximal intersecting family of 3-sets, that is no other subset of three points could be added such that the family remains intersecting. Compare with the maximum in the Erdős-Ko-Rado theorem.

8-3: The upper bound $\binom{n-1}{k-1}$ given by the Erdős–Ko–Rado theorem is achieved by families of sets containing a fixed element. Show that for $n = 2k$, there are other families achieving this bound. *Hint:* Give a large intersecting family containing k-element sets only.

8-4: Let $k < \frac{n}{2}$. Let X_k denote the family of k-element subsets of $[n]$. Define the following bipartite graph on the vertex set $X_k \cup X_{k+1}$. Two sets are connected by an edge if, and only if, one contains the other. First, show that there is a matching M in this graph such that each member of X_k is covered by an edge of M. Second, use this fact to prove Sperner's theorem.

8-5: Let $\mathcal{F}$ be a family of at least $k + 1$ subsets of $\{1, \ldots, n\}$ each containing at most k elements. Prove that if any $k+1$ members of $\mathcal{F}$ share a common element then all of them do.

8-6: * Let 71356 . . . be an infinite sequence of digits defined as follows. Each digit starting with the fifth one is the last digit of the sum of the previous four digits in the sequence. Does the sequence 4713 appear in our infinite sequence?