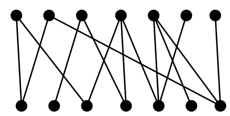
Graph theory - solutions to problem set 8

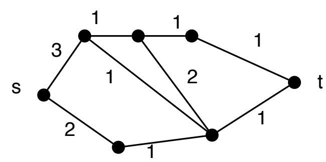
Exercises

1. Find a minimum vertex cover in the following graph.



   Solution: Since this graph has a perfect matching, the vertices from the top part form a minimum vertex cover.

2. Find a maximum flow from s to t and a minimum s-t cut in the following network.

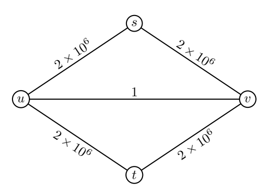


   Solution: One possibility for a maximum flow (of total value 2) is to push a flow of size 1 through the path formed by edges on the top, and another one of size 1 through the edges in the bottom. The used capacities on the edges are then 1/3, 1/1, 1/1, 1/1, 0/1, 0/2, 1/2, 1/1, 1/1.

   An example of a minimum cut (of capacity 2) is $(V - t, t)$.

3. Construct a network on four vertices for which the Ford-Fulkerson algorithm may need more than a million iterations, depending on the choice of augmenting paths.

   Solution: Consider the following network:



   Clearly, the maximum value of a flow is 2 million, but the Ford-Fulkerson algorithm might always choose a path that uses the edge uv by alternately choosing $suvt$ and $svut$. This way the value of the flow increases by 1 in every step, so it takes 2 million improvements to reach a maximum flow.

4. Let G be a network with source s, sink t, and integer capacities. Prove or disprove the following statements:

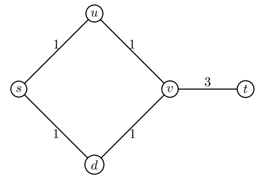
   (a) If all capacities are even then there is a maximal flow f such that $f(e)$ is even for all edges e.

   (b) If all capacities are odd then there is a maximal flow f such that $f(e)$ is odd for all edges e.

   Solution:

   (a) Dividing all capacities by two, we obtain a network with integral capacities. Therefore, it has an integral maximum flow. Multiplying this flow by 2, we get an even maximum flow on the original network.

   (b) Counter example:



Problems

5. Let G be a graph on n vertices

   (a) Show that $\tau(G) + \alpha(G) = n$.

   (b) Show that if G is bipartite then $\nu(G) + \alpha(G) = n$.

   Solution:

   (a) First note that $X \subseteq V(G)$ is a vertex cover if and only if $V(G) \setminus X$ is an independent vertex set. To prove the case $\Rightarrow$, suppose by the contradiction that there exist vertices $u, v \in V(G) \setminus X$ such that $uv \in E(G)$. Then the edge uv is not covered by any of the vertices in X, which is in contradiction with the assumption that X is a vertex cover. One can prove the $\Leftarrow$ case by a similar reasoning.

   Now take the vertex cover X with $|X| = \tau(G)$. We know $V(G) \setminus X$ is an independent vertex set, so we have
   $$n = |X| + |V(G) \setminus X| = \tau(G) + |V(G) \setminus X| \leq \tau(G) + \alpha(G).$$
   On the other hand, consider the independent vertex set Y with $|Y| = \alpha(G)$. By the remark, $V(G) \setminus Y$ forms a vertex cover.
   $$n = |Y| + |V(G) \setminus Y| = \alpha(G) + |V(G) \setminus Y| \geq \alpha(G) + \tau(G).$$
   Therefore, we have $\tau(G) + \alpha(G) = n$.

   (b) It follows from part (a) and the Kőnig's Theorem ($\nu(G) = \tau(G)$).

6. Let G be a network with source s, sink t, and integer capacities. Prove that an edge e is saturated (i.e., the flow uses its full capacity) in every maximum s-t flow if and only if decreasing the capacity of e by 1 would decrease the maximum capacity of an s-t flow in G.

   Solution: Let k be the value of the maximum flow, i.e., the capacity of a minimum cut in G. We argue by contradiction for both directions.

   $\Rightarrow$: Suppose decreasing the capacity does not decrease the size of the max flow. Then there is a flow in this new network G' of value k that does not use the full capacity of e in G. This corresponds to a flow of size k in G with the same property, contradicting our assumption.

   $\Leftarrow$: Suppose e is not saturated in every maximum s-t flow. If we could assume that this flow is an integer flow, then we would be done: This flow would have capacity k in G', contradicting the assumption. But we don't quite know that we can assume this, so let's do something else instead:

   Indeed, if e is not saturated in some maximum flow, then e does not occur in any min cut (otherwise the size of this flow would be less than the capacity of a min cut). Now we do know that every cut has an integer capacity, so if e is not in a k-cut, then every cut containing e has capacity at least $k + 1$. Hence decreasing the capacity of e by 1 will not create any cut of size k, therefore the size of the maximum flow does not decrease. Contradiction.

7. Deduce Hall's theorem from the max-flow min-cut theorem.

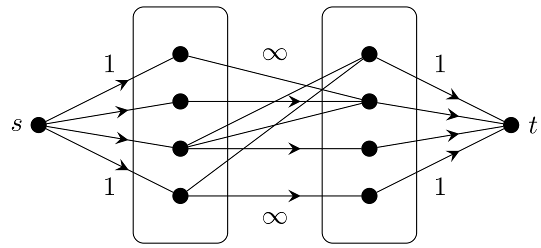
   Solution: Consider the bipartite graph $G = (A \cup B, E)$ that satisfies Hall's conditions. Let us make a network out of G. Add a source s, connect it to all vertices of A by oriented edges of capacity 1. Analogously, add a sink t and connect all vertices of B to it by edges of capacity 1. Let the edges of

G be oriented from A to B and their capacities to be infinite. We stress that all the edges are now oriented in the direction from s to t (see the figure below).

If there is a integer flow of value $|A|$ in G, then the edges (x, y) satisfying $x \in A, y \in B$, and $f(x, y) = 1$ constitute a matching of A in G, and we are done. Otherwise, there is a cut (X, Y) of capacity $k < |A|$. We know that

$$|A \cap Y| + |B \cap X| = k < |A| = |A \cap X| + |A \cap Y|,$$

from which we conclude that $|B \cap X| < |A \cap X|$. Let $W = A \cap X$. The set $N(W)$ is contained in $B \cap X$, as otherwise there would be an infinite-capacity edge crossing from X to Y. Thus, $|N(W)| = |B \cap X| < |W|$, and we verified that when a perfect matching does not exist, there is a set W violating Hall's criterion.

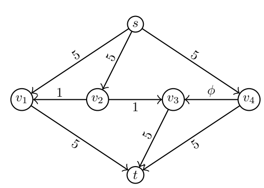


8. Let A be an $n \times m$ matrix of non-negative real numbers such that the sum of the entries is an integer in every row and in every column. Prove that there is an $n \times m$ matrix B of non-negative integers with the same sums as in A, in every row and every column.

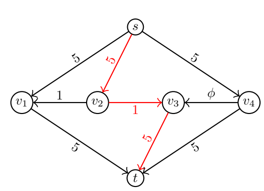
   Solution: Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ be the sums of the entries in the rows and columns of A, respectively. Consider the complete bipartite graph $G = (P \cup Q, E)$, where $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_m\}$. Let us make a directed network out of G. Orient all edges from P to Q, and set their capacity to infinity. Add a source s, connect it to all vertices of P by oriented edges of capacity a_i for the vertex p_i. Analogously, add a sink t and connect all vertices of Q to it by oriented edges of capacity b_j for the vertex q_j.

   Clearly, the minimum capacity of a cut is $a_1 + \cdots + a_n = b_1 + \cdots + b_m$. Since all capacities are integral, the Ford-Furkerson theorem shows that there is an integer flow with maximum value. We can then define the entry B_{ij} of the matrix B to be the value of this flow on the edge $p_i q_j$.

9. Prove that the Ford-Fulkerson algorithm might not stop on the following network with the capacities shown on the edges, where $\phi = \frac{\sqrt{5}-1}{2}$. For this, use induction to show that the "residual capacities" $c(u, v) - f(u, v)$ on the three horizontal edges can be $\phi^k, 0, \phi^{k+1}$, for every k. (Note that $\phi^2 = 1 - \phi$.)



   Solution: Suppose the Ford-Fulkerson algorithm starts by choosing the central augmenting path, shown in the figure below.



The three horizontal edges, in order from left to right, now have residual capacities 1, 0, and ϕ. Suppose inductively that the horizontal residual capacities are ϕ^{k-1}, 0, ϕ^k for some positive integer k.

1. Augment along B, adding ϕ^k to the flow; the residual capacities are now $\phi^{k+1}, \phi^k, 0$.
2. Augment along C, adding ϕ^k to the flow; the residual capacities are now $\phi^{k+1}, 0, \phi^k$.
3. Augment along B, adding ϕ^{k+1} to the flow; the residual capacities are now $0, \phi^{k+1}, \phi^{k+2}$.
4. Augment along A, adding ϕ^{k+1} to the flow; the residual capacities are now $\phi^{k+1}, 0, \phi^{k+2}$.