Graph theory - solutions to problem set 2

Exercises

1. Prove the triangle-inequality in graphs: for any three vertices u, v, w in a graph G,

$$d(u,v) + d(v,w) \geq d(u,w).$$

Solution. If $d(u,v) = \infty$ or $d(v,w) = \infty$, there is nothing to prove.

Otherwise, according to the definition of the distance, there is a u-v path of the length $d(u,v)$ and a v-w path of the length $d(v,w)$. Joining them together we obtain the u-w walk of the length $d(u,v)+d(v,w)$. We have seen in class, that this walk will then contain a u-w walk, which is clearly not longer than the walk. Therefore, the shortest u-w path is no longer than $d(u,v) + d(v,w)$.

2. Show that a graph is connected if and only if it contains a spanning tree.

Solution. If there is a spanning tree then the graph is clearly connected: for any vertices u and v, there will be a u-v path in the tree, hence in the graph, as well. If the graph is connected then the BFS algorithm finds a spanning tree, and this proves that a spanning tree exists.

3. Prove that a forest on n vertices with c connected components has exactly $n - c$ edges.

Solution. Let $T_1, \ldots, T_c$ be the components of the forest, on $n_1, \ldots, n_c$ vertices, respectively. Each T_i itself is a connected acyclic graph, hence it is a tree (considered as a graph on its own). Therefore, T_i contains $n_i - 1$ edges for each i. Altogether, the graph contains $\sum_{i=1}^{c}(n_i - 1) = \sum_{i=1}^{c} n_i - c = n - c$ edges.

4. Let T be a tree and e be an edge of T. Prove that $T - e$ is not connected.

Solution. Let $e = uv$ and suppose $T - e$ is connected. Then, in particular, $T - e$ contains a u-v path P. But then $P + e$ is a cycle in T, a contradiction.

5. Let T be a tree and let u and v be two non-adjacent vertices of T. Prove that $T + uv$ contains a unique cycle.

Solution. T contains a u-v path and adding uv to it will form a cycle, so $T + uv$ contains at least one cycle. Suppose it has two different cycles. Both of them must contain uv, otherwise, removing uv, we would get a cycle in T. But then if we remove uv, we get two different u-v paths in T, which contradicts a result from the lecture. Hence there is a unique cycle.

Problems

6. Let G be a graph on n vertices. Prove that

   (a) if G has dn edges, then it contains a path of length at least d.

   (b) if G has at least $2n - 1$ edges, then it contains an even cycle.

Solution.

(a) On the lecture we proved that G contains a subgraph H with $\delta(H) > d$. By another result from the lecture, H contains a path of length at least $\delta(H)$.

(b) On the lecture we proved that G contains a bipartite subgraph H with $|E(H)| \geqslant \frac{|E(G)|}{2}$. We have that $|V(H)| \leq n$ and $|E(H)| \geq n$ in H. Therefore, H contains a cycle (otherwise H is a forest, and the number of edges in a forest is strictly less than the number of vertices). Since H is a bipartite graph, this cycle has even length.

7. Let W be a closed walk that uses the edge e exactly once. Prove that W contains a cycle through e.

   Solution. Let $v_1 v_2 \dots v_n v_1$ be a shortest closed walk that uses the edge e exactly once. We claim that this walk is a cycle. Indeed, if $v_i = v_j$ for some $i < j$, then either the closed walk $v_1 \dots v_i v_{j+1} \dots v_1$ or the closed walk $v_i v_{i+1} \dots v_j$ uses the edge e exactly once, and both of them are shorter, which is not possible. (Why doesn't this argument work for an arbitrary walk that uses the edge e exactly twice?)

8. Prove that every connected graph on $n \geq 2$ vertices has a vertex that can be removed without disconnecting the remaining graph.

   Solution. Take a spanning tree T of the graph. It has at least two leaves, say x and y. Then $T - x$ and $T - y$ are both connected, hence so are their supergraphs, $G - x$ and $G - y$.

9. Let T be a tree on n vertices that has no vertex of degree 2. Show that T has more than $n/2$ leaves.

   Solution. T has $n - 1$ edges, so the sum of the degrees is $2n - 2$. Suppose T has at most $n/2$ leaves. Then at least $n/2$ vertices have degree at least 3. But then the sum of the degrees is at least $1 \cdot \frac{n}{2} + 3 \cdot \frac{n}{2} = 2n$, which is a contradiction.

10. Show that every tree T has at least $\Delta(T)$ leaves.

    Solution. Let v be a vertex with degree $d = \Delta(T)$. For every edge vw incident to v, take a longest path starting with vw. By maximality (as in the proof that every tree has a leaf), the last vertex of this path is a leaf. Doing this for each of the d edges incident to v, we get d paths starting at v, which are disjoint except for v (otherwise we would get a cycle). Thus each path gives a different leaf, and we get $d = \Delta(T)$ leaves.

    Alternative solution: If you remove v and its incident edges, you are left with d connected components $T_1, \dots, T_d$, each of which is a tree. By a lemma from class, every tree with at least two vertices has at least two leaves. Hence the T_i with at least two vertices have at least two leaves, one of which must be a leaf of T (one of the two leaves might have been adjacent to v, but not both because that would give a cycle). Some of the T_i might be single vertices, in which case those vertices were leaves in T (they must have been adjacent to v and to no other vertex).

11. Let T be an n-vertex tree that has exactly $2k$ vertices of odd degree. Show that T can be split into k edge-disjoint paths (i.e., T is the union of k edge-disjoint paths).

    Solution. We prove a more general statement: the above claim is true for forests, not only trees. Making our problem more general allows us to use a simpler induction argument.

    So let us do induction on k. For $k = 0$ our forest is empty (every nonempty forest has a leaf, thus an odd-degree vertex), so the statement holds. Now assume we know it for k, and take a forest T with $2k + 2$ odd-degree vertices. Let P be a maximal path in T. We have seen in class that P will connect two leaves. We claim that if we delete the edges of P then we get a forest $T - P$ with $2k$ odd-degree vertices. Indeed, the two leaves will lose the edge touching them, so they have degree 0 in $T - P$, while every other vertex loses either 0 or 2 incident edges, hence the parity of its degree does not change. In other words, we lost two odd-degree vertices and did not gain anything. So we can apply induction on $T - P$ to get k paths partitioning its edge set. Together with P we have $k + 1$ paths partitioning the edge set of T, which is what we wanted to show.