Graph theory - solutions to problem set 10

Exercises

1. Prove that if G is a K_3-free graph, then $\alpha(G) \geq \Delta(G)$.

   Solution: If G is triangle-free then every neighborhood is an independent set, so $\alpha(G) \geq \Delta(G)$.

2. Prove the lower bound for the Erdős-Stone-Simonovits theorem, i.e., for every graph H with chromatic number $s \geq 2$, $\text{ex}(n, H) \geq |E(T(n, s-1))|$.

   Solution: The Turán graph $T(n, s-1)$ is $(s-1)$-colorable, so it does not contain any subgraph of chromatic number s, in particular, $\text{ex}(n, H) \geq |E(T(n, s-1))|$.

3. (a) Deduce from the proof of Mantel's theorem that $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is the only "extremal" K_3-free graph, i.e., every K_3-free graph with $\text{ex}(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$ edges is isomorphic to G.

   (b) Deduce from the proof of Turán's theorem that $T(n, r)$ is the only extremal K_{r+1}-free graph.

   Solution:

   (a) Analyzing the proof, we see that the optimum is reached when $\Delta = \lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$, and every vertex not adjacent to v touches exactly Δ edges, which are all distinct. This is only possible if there is no edge going between two non-neighbors of v, i.e., the non-neighbors are all connected to the neighborhood of v. Then the graph is isomorphic to $K_{\Delta, n-\Delta}$.

   (b) See lecture notes.

4. Let L be a set of n lines in the plane and P a set of n points in the plane. Prove that the number of point-line incidences, i.e., pairs $(p, \ell) \in P \times L$ with $p \in \ell$ is $O(n^{3/2})$.

   Solution: Let G be the bipartite graph on $2n$ vertices with parts P and L as vertices, where a point in P is connected to a line in L if the point lies on the line. Note that this graph is $K_{2,2}$-free. Indeed, two lines in L share at most 1 point in common. Then we can apply the Kővári-Sós-Turán theorem on G to see that the graph has at most $cn^{3/2}$ edges for some constant c.

Problems

5. Recall from problem set 8 that $\alpha(G) + \tau(G) = |V(G)|$. Prove that if G is triangle-free then $|E(G)| \leq \alpha(G) \cdot \tau(G)$, and use this to reprove Mantel's Theorem.

   Solution: By Exercise 1, we have $\alpha(G) \geq \Delta(G)$. Now take a vertex cover of size $\tau(G)$. They touch all edges. On the other hand, the number of edges touching these vertices is at most $\tau(G)\Delta(G) \leq \tau(G)\alpha(G)$, proving the first statement. The second statement is just Cauchy-Schwarz (or AM-GM): if $x + y = n$ then $xy \leq n^2/4$.

6. Let P be a set of n points in $\mathbb{R}^2$, such that no two points are more than distance 1 apart. Show that there are at most $n^2/3$ pairs of points whose distance is greater than $1/\sqrt{2}$.

   [Hint: .seerged 90 tsael ta elgna na htiw elgnairt a si ereht stniop ruof gnoma taht wohS]

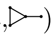
   Solution: Observe, first, that among any 4 points in the plane some three form a non-acute triangle (with an angle of at least 90 degrees). Indeed, if the points are in convex position, then one of the four angles of the quadrilateral is at least 90 degrees (their sum is 360), whereas if their convex hull is a triangle, then one (even two, in fact) of the three angles at the point in the middle is at least 90 degrees (again, their sum is 360).

Now look at the set P, and draw an edge between two points if their distance is greater than $1/\sqrt{2}$. We claim that this graph contains no K_4. Suppose it did, and look at the four points inducing a K_4. By the above claim, three of them form a non-acute triangle. Let us denote the sides by a, b, c, where the angle between a and b is at least 90 degrees. Both a and b have length greater than $1/\sqrt{2}$, so c must have length greater than 1, contradiction.

So this graph we defined is K_4-free, hence, by Turán's theorem, it contains at most $n^2/3$ edges, what we wanted to show.

7. Let G be a d-regular graph on n vertices with girth at least $2k+1$. Prove that $d \leq n^{1/k}$, i.e., G has at most $\frac{1}{2}n^{1+1/k}$ edges.

   Solution: Suppose $d > n^{1/k}$. Let v be a vertex and let $V_1, \ldots, V_k$ be subsets of vertices where V_i is the set of vertices of distance i from v. As there is no cycle of length less than $2k+1$, no vertex in V_i is connected to multiple vertices in V_{i-1}. The regularity then implies $|V_i| = d(d-1)^{i-1}$. But then G contains at least $|V_k| \geq d(d-1)^{k-1} > (d-1)^k \geq n$, a contradiction.

8. Show that ex(n, ) $= \lfloor \frac{n^2}{4} \rfloor$ for every $n > 3$.

   Solution: One way to prove this is to use Mantel's theorem: We know there is a triangle in any graph with more than $\lfloor \frac{n^2}{4} \rfloor$ edges. This forms a graph in question, unless this triangle is isolated. One can then delete the vertices of the triangle and iterate on the remaining, denser graph. Eventually this will yield a graph that is way too dense not to contain the graph. (fyi: this was just a solution sketch).

   Here we will show a different way that repeats the argument of Mantel. Take a vertex v of max degree. If v has degree at least 3, then its neighborhood contains no edge: otherwise we get a graph we need. We can then repeat the proof of Mantel to get that the number of edges is at most $\Delta(n-\Delta) \leq \lfloor \frac{n^2}{4} \rfloor$. Otherwise, $\Delta \leq 2$, so the total number of edges is at most $n\Delta/2 = n$. When $n > 3$, this is at most $\lfloor \frac{n^2}{4} \rfloor$, so we are done.

9. This exercise is about constructing a $K_{2,2}$-free graph on n vertices with $n^{3/2}$ edges for large n.

   Let $p \geq 3$ be a prime, and G_0 be the graph on the vertex set $\mathbb{Z}_p \times \mathbb{Z}_p$ where (x, y) and (x_1, y_1) are connected by an edge and only if $x + x_1 = yy_1$. (Technically this is a multigraph as it has loops.) Let G be the graph on $n = p^2$ vertices that we get by deleting the loops from G_0.

   (a) Prove that G_0 is p-regular and has at most p loops.

   (b) Deduce that G has $(\frac{1}{2} + o(1))n^{3/2}$ edges.

   (c) Show that any two vertices in G have at most 1 common neighbor (and hence G is $K_{2,2}$-free).

   Solution: See lecture notes.