Graph theory - solutions to problem set 1

Exercises

1. (a) Is C_n a subgraph of K_n?
   (b) For what values of n and m is $K_{n,n}$ a subgraph of K_m?
   (c) For what n is C_n a subgraph of $K_{n,n}$?

   Solution:

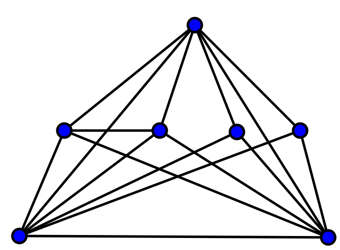
   (a) Yes! (you can check it by the definition of the subgraph given in the lecture, or just simply by the fact that K_n has all the possible edges a graph on n vertices can have.)
   (b) We must have $m = |V(K_m)| \geq |V(K_{n,n})| = 2n$. On the other hand, by a similar reasoning as part (a), we get that the statement holds for all m, n with $m \geq 2n$.
   (c) First, note that a bipartite graph cannot have any cycle of odd length, so n cannot be odd. For even n, one can check that $K_{n,n}$ has a cycle of length n.

2. Given a graph G with vertex set $V = \{v_1, \ldots, v_n\}$ we define the *degree sequence* of G to be the list $d(v_1), \ldots, d(v_n)$ of degrees in decreasing order. For each of the following lists, give an example of a graph with such a degree sequence or prove that no such graph exists:

   (a) $3, 3, 2, 2, 2, 1$
   (b) $6, 6, 6, 4, 4, 3, 3$

   Solution:

   (a) There is no such graph; since by problem 5, the number of odd-degree vertices in a graph is always even.
   (b) Consider the following graph:



3. Construct two graphs that have the same degree sequence but are not isomorphic.

   Solution: Let G_1 be of a cycle on 6 vertices, and let G_2 be the union of two disjoint cycles on 3 vertices each. In both graphs each vertex has degree 2, but the graphs are not isomorphic, since one is connected and the other is not.

4. A graph is *k-regular* if every vertex has degree k. How do 1-regular graphs look like? And 2-regular graphs?

   Solution: A 1-regular graph is just a disjoint union of edges (soon to be called a matching). A 2-regular graph is a disjoint union of cycles.

5. Prove that the number of odd-degree vertices in a graph is always even. **Solution:** Let $G = (V, E)$ be an arbitrary graph. In the lecture we have proved that $\sum_{v\in V} d(v) = 2|E|$.

   Let $V_1 \subseteq V$ be the set of vertices of G which have odd degree and $V_2 = V\backslash V_1$ be the set of vertices of G which have even degree. We have that

   $$\sum_{v\in V} d(v) = \sum_{v\in V_1} d(v) + \sum_{v\in V_2} d(v) = 2|E|.$$

   Since all the vertices in V_2 have even degree, and $2|E|$ is even, we obtain that $\sum_{v\in V_1} d(v)$ is even. But since V_1 is the set of vertices of odd degree, we obtain that the cardinality of V_1 is even (that is, there are an even number of vertices of odd degree), which completes the proof.

6. How many (labelled) graphs exist on a given set of n vertices? How many of them contain exactly m edges?

   Solution: Since there are $\binom{n}{2}$ possible edges on n vertices, and a graph may or may not have each of these edges, we get that there are $2^{\binom{n}{2}}$ possible graphs on n vertices. For the second problem, out of the $\binom{n}{2}$ possible edges, we want to choose m ones. So there are $\binom{\binom{n}{2}}{m}$ possible graphs on n vertices and with m edges.

Problems

7. Do graphs with the following degree sequences exist:

   (a) $6, 6, 6, 4, 4, 2, 2$

   (b) $6, 6, 6, 6, 5, 4, 2, 1$?

   Solution:

   (a) No, since otherwise we have 3 vertices of degree 6 which are adjacent to all other vertices of the graph; so each vertex in the graph must be of degree at least 3.

   (b) No! Note that each vertex of the degree 6 is adjacent to all but one other vertices. In particular, each such vertex is adjacent to at least one of v_1 and v_2 (where $d(v_1) = 1$ and $d(v_2) = 2$). But that would mean at least four edges touching v_1 or v_2, contradicting $d(v_1) + d(v_2) = 3 < 4$.

8. Let G be a graph with minimum degree $\delta > 1$. Prove that G contains a cycle of length at least $\delta + 1$. **Solution:** First, let's recall how we proceeded in the lecture to find a path of length at least δ:

   Let $v_1 \cdots v_k$ be a maximal path in G, i.e., a path that cannot be extended. Then any neighbor of v_1 must be on the path, since otherwise we could extend it. Since v_1 has at least $\delta(G)$ neighbors, the set $\{v_2, \ldots, v_k\}$ must contain at least $\delta(G)$ elements. Hence $k \geq \delta(G) + 1$, so the path has length at least $\delta(G)$.

   Now in order to find a cycle of length at least $\delta + 1$, we continue the proof above. The neighbor of v_1 that is furthest along the path must be v_i with $i \geq \delta(G) + 1$. Then $v_1 \cdots v_i v_1$ is a cycle of length at least $\delta(G) + 1$.

9. How many (labelled) graphs on the vertex set $\{1, \ldots, n\}$ are isomorphic to P_n? How many are isomorphic to C_n?

   Solution: Consider the problem for P_n. Note that each such a graph is of the form $\sigma(1)\sigma(2)\ldots\sigma(n)$ for some permutation σ of $\{1, 2, \ldots, n\}$. On the other hand, the graphs formed by $\sigma(1)\sigma(2)\ldots\sigma(n)$ and $\sigma(n)\sigma(n-1)\ldots\sigma(1)$ are just the same. (Check the figure for $n = 5$) Since there are $n!$ permutations in total, we get $n!/2$ distinct labeled graphs which are isomorphic to P_n.

   For C_n, the following forms all give the same graph for a fixed permutation σ: (Check the figure for $n = 4$)

   $$\sigma(1)\sigma(2)\ldots\sigma(n)\sigma(1), \sigma(2)\sigma(3)\ldots\sigma(1)\sigma(2), \ldots, \sigma(n)\sigma(1)\ldots\sigma(n-1)\sigma(n),$$
   $$\sigma(n)\sigma(n-1)\ldots\sigma(1)\sigma(n), \sigma(1)\sigma(n)\ldots\sigma(2)\sigma(1), \ldots, \sigma(n-1)\sigma(n-2)\ldots\sigma(n)\sigma(n-1),$$

   So we get $\frac{n!}{2n} = \frac{(n-1)!}{2}$ distinct labeled graphs which are isomorphic to C_n.

Figure 1: these two graphs are the same

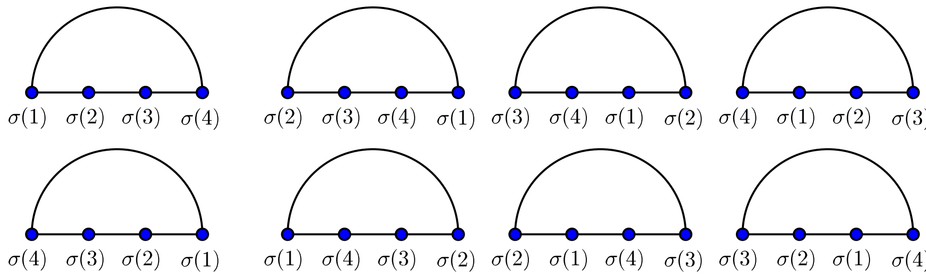


Figure 2: these 8 graphs are all the same

10. Show that every graph on at least two vertices contains two vertices of equal degree.

    Solution: Suppose that the n vertices all have different degrees, and look at the set of degrees. Since the degree of a vertex is at most $n-1$, the set of degrees must be

    $$\{0, 1, 2, \ldots, n-2, n-1\}.$$

    But that's not possible, because the vertex with degree $n-1$ would have to be adjacent to all other vertices, whereas the one with degree 0 is not adjacent to any vertex.

11. What is the maximum number of edges in a bipartite graph on n vertices? (Prove your answer.)

    Solution: Let $G = (A \cup B, E)$ be a bipartite graph, with A, B disjoint and $|A| + |B| = n$. Since all the edges of G have one endpoint in A and the other in B, the number of edges $|E|$ of G cannot exceed the number of pairs $(a, b) \in A \times B$, so $|E| \leq |A| \cdot |B| = |A|(n - |A|)$. Intuitively, such a product is maximized when the two factors are equal, so when $|A| = \lfloor n/2 \rfloor$. More formally, we can use the inequality $4xy \leq (x+y)^2$ to get

    $$|E| \leq |A|(n - |A|) \leq \frac{(|A| + n - |A|)^2}{4} = \frac{n^2}{4}.$$

    Therefore, the number of edges of a bipartite graph on n edges is at most $n^2/4$.

    Note that $n^2/4$ is exactly the maximum when n is even, because then it is attained by the complete bipartite graph $K_{n/2,n/2}$. When n is odd, the maximum is actually $\lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil = \frac{n^2-1}{4}$, which is attained by $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.