

## Solutions to the midterm

**Exercise 1 :** (9 points: 3 for each part)

Answer the following 5 questions (give the definitions).

a. What is a tree?

**Solution.** *A tree is a connected graph without cycles.*

b. What is the chromatic number of a graph?

**Solution.** *The chromatic number of a graph is the minimum number of colors one can use to color the vertices of the graph, such that no two adjacent vertices have the same color.*

c. What is a matching in a graph?

**Solution.** *A matching in a graph is a set of pairwise non-adjacent edges (that is, no two edges have a common vertex).*

**Exercise 2 :** (21 points: 3 for each part)

For each of following 7 statements decide whether they are true or false.

**Justify your answers. You may refer to the theorems from the course.**

a. Any bipartite graph has a perfect matching.

**Solution.** *False*

*Consider for instance the graph  $G = (V, E)$ , where  $V = \{1, 2, 3\}$ , and  $E = \{(1, 2), (1, 3)\}$ .  $G$  is bipartite with bipartition  $A = \{1\}$ ,  $B = \{2, 3\}$ , but does not have a perfect matching.*

b. If a graph is  $k$ -connected then it is also  $k$ -edge connected.

**Solution.** *True*

*We know (by Menger's theorem) that:*

*- a graph is  $k$ -connected if and only if there are at least  $k$  paths between any two vertices such that no two share any internal points*

- a graph is  $k$ -edge connected if and only if there are at least  $k$  paths between any two vertices such that no two share any edges.

It is clear that if two paths do not share any internal points, then for sure they do not share any edge, so every  $k$ -connected graph is also  $k$ -edge connected.

c. In any poset any two elements are comparable.

**Solution.** *False.*

**Counter-example 1** Consider for instance the power set of the set  $X = \{1, 2\}$ , together with the containment relation. This forms a poset, but for instance the elements  $\{1\}$  and  $\{2\}$  of the poset are not comparable (no one is contained in the other).

**Counter-example 2** The positive integers, together with the divisibility relation form a poset. In this poset, the elements 2 and 3 are not comparable.

d. The capacity of any cut in any network is less than the maximum flow in the network.

**Solution.** *False.*

We know that in a network, the capacity of a minimum cut equals the maximum flow. Therefore, the capacity of any cut is at least as big as the maximum flow.

e. The line graph of any planar graph is planar.

**Solution.** *False.*

Consider for instance the planar graph  $G = (V, E)$ , where  $V = \{1, 2, 3, 4, 5, 6\}$ , and  $E = \{(1, 2), (1, 3), (1, 4), (1, 5), (1, 6)\}$ . The line graph of  $G$  is isomorphic to  $K_5$ , so it is not planar.

f. Any graph with chromatic number 2 is bipartite.

**Solution.** *True.*

Since the chromatic number is 2, then we can color the vertices with two colors (red and blue) such that there is no edge between any two vertices having the same color. Let  $A$  be the set of the red vertices, and  $B$  the set of the blue vertices. This gives already a bipartition of the graph, since there are no edges between vertices from the same set.

g. If  $G$  contains  $K_5$  as a topological minor then it is not planar.

**Solution.** *True.*

This follows from Kuratowski's theorem.

**Exercise 3 :** (20 points: 5 for each problem.)

**Justify your answers. You may refer to the theorems from the course.**

a. Draw a tree corresponding to the Prüfer code (533314).

**Solution.** *Since the Prüfer code has length 6, then the tree has 8 vertices.*

*The tree is given by  $T = (V, E)$ , where  $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$ , and  $E = \{(2, 5), (5, 3), (3, 6), (3, 7), (1, 3), (4, 1), (4, 8)\}$ .*

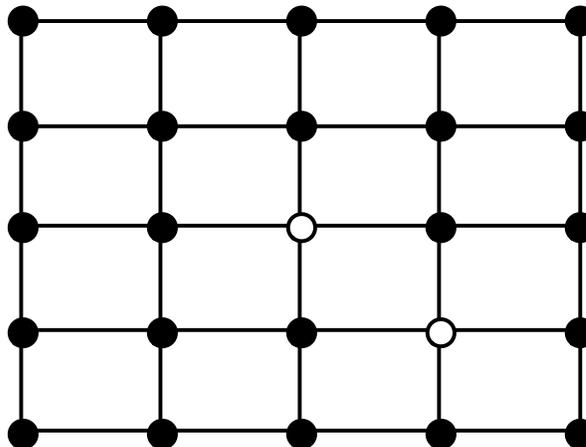
b. What is the independence number of the graph with vertices  $\{1, 2, 3, 4, 5, 6, 7\}$  and with the edge set  $\{\{1, 2\}, \{1, 5\}, \{1, 3\}, \{4, 3\}, \{2, 6\}, \{5, 7\}\}$ ?

**Solution.** *The independence number of the graph is 4. In order to see this, just pick the vertices  $\{1, 4, 6, 7\}$ . No two of them are connected. Note that any pair set of 5 vertices will contain at least one edge.*

c. How many different spanning trees there are in a cycle on 23 vertices?

**Solution.** *Note that the removal of any edge from the cycle leaves us with a spanning tree (and every spanning tree of the cycle can be obtained in this way). Since there are 23 edges in the cycle, and for every removal of an edge we obtain a different spanning tree, then the number of different spanning trees is exactly 23.*

d. Consider a network on the graph of the grid that is given on the picture. Let all the capacities of edges of the graph be unit. What is the max-flow between the two white vertices?



**Solution.** *We know that the capacity of a minimum cut equals the maximum flow. Therefore, it is enough to find a minimum cut. Note that removing all the edges incident to one of the two white vertices gives a minimum cut, which has capacity 4. On the other hand we note that it is easy to find 4 edge disjoint paths between the given vertices. Hence, the maximum flow will be 4.*

*You can also use Ford-Fulkerson algorithm.*

**Exercise 4 :** (10 points.)

Show that if a graph does not have  $K_3$  as a topological a minor then it has at most  $n - 1$  edges.

**Solution.** *Let  $G$  be the graph in the problem. Since  $G$  does not contain  $K_3$  as topological minor, then it cannot contain any cycle. Indeed, assuming that  $G$  contains a cycle, we could delete all the vertices and edges outside the cycle, and contract all but two edges in the cycle, obtaining  $K_3$  (which is clearly a contradiction).*

*Therefore,  $G$  is a forest (a union of trees), so we can split it in connected components (which are trees)  $T_1 = (V_1, E_1), \dots, T_k = (V_k, E_k)$ , where  $|V_1| + |V_2| + \dots + |V_k| = n$  and  $|E_1| + \dots + |E_k| = |E|$ . Since every  $T_i$  is a tree, we have that  $|E_i| = |V_i| - 1$ , for every  $1 \leq i \leq k$ . Therefore, we get that*

$$|E| = |E_1| + \dots + |E_k| = |V_1| - 1 + \dots + |V_k| - 1 = n - k \leq n - 1,$$

*which finishes the proof.*

**Exercise 5 :** (10 points.)

Prove that the number of labeled connected graphs on  $n$  vertices that contain at most one cycle is at most  $n^n$ .

**Solution.** *Let us observe that every labeled connected graph on  $n$  vertices having at most one cycle (call this set of graphs  $G(n)$ ) is either a tree or contains exactly one cycle. Denoting by  $T(n)$  the number of labeled trees on  $n$  vertices, and by  $C(n)$  the number of connected graphs containing exactly one cycle, we have that  $G(n) = T(n) + C(n)$ .*

*By Cayley's theorem, we know that  $T(n) = n^{n-2}$ . On the other hand, every connected graph containing exactly one cycle can be obtained from a tree by adding exactly one edge. Since the number of possible edges in a graph on  $n$  vertices is  $\binom{n}{2}$ , for every tree there are exactly  $\binom{n}{2} - (n - 1)$  different ways of adding a new edge (you subtract from the total number of possible edges the number of edges that are already in the graph). Since there are exactly  $n^{n-2}$  labeled trees on  $n$  vertices, we obtain that*

$$C(n) \leq (n(n-1)/2 - n + 1) n^{n-2}.$$

*This implies that*

$$G(n) = T(n) + C(n) \leq n^{n-2} + (n(n-1)/2 - n + 1) n^{n-2} \leq n^{n-2} + n(n-1)n^{n-2} \leq n^n,$$

*which completes the proof.*

**Exercise 6 :** (15 points.)

Consider a bipartite graph  $G = (A \cup B, E)$ ,  $|A| = |B| = 7$ , with vertices in each part numbered from 1 to 7. Let a vertex  $i \in A$  be connected to a vertex  $j \in B$  iff  $|i - j|$  is a square of an integer modulo 7. Does this graph have a perfect matching?

**Solution.** Note that  $|i - i| = 0 = 0^2 \pmod{7}$ , for every  $i \in \{1, \dots, 7\}$ , so every vertex with label  $i$  from  $A$  is connected to the vertex with label  $i$  from  $B$ . This gives you right away a perfect matching.

There are also other perfect matchings. For example, you might pick the edges  $(1, 5), (2, 6), (3, 7), (4, 4), (5, 1), (6, 2), (7, 3)$ , where the first vertex in each pair is taken from  $A$ , and the second is taken from  $B$ . Indeed, the pairs we have picked are indeed edges of the graph, since we have that  $|1 - 5| = |2 - 6| = |3 - 7| = |5 - 1| = |6 - 2| = |7 - 3| = 4 = 2^2 \pmod{7}$ , and  $|4 - 4| = 0 = 0^2 \pmod{7}$ .

**Exercise 7 :** (15 points.)

Prove that the number of labeled connected graphs on 10 vertices is strictly bigger than the number of labeled disconnected graphs on 10 vertices.

**Solution.** Let us denote by  $\mathcal{D}$  the set of labeled disconnected graphs on 10 vertices, and by  $\mathcal{C}$  the set of labeled connected graphs on 10 vertices. We have to show that  $|\mathcal{D}| < |\mathcal{C}|$ , where  $|\mathcal{D}|$ , and respectively  $|\mathcal{C}|$ , denote the cardinality of  $\mathcal{D}$ , respectively  $\mathcal{C}$ . We start by proving that  $|\mathcal{D}| \leq |\mathcal{C}|$  by establishing an injective function  $f : \mathcal{D} \rightarrow \mathcal{C}$ , which will associate to every disconnected graph  $D$  a unique connected graph  $G$  a unique connected graph.

We will prove that the complement of every disconnected graph  $D$  is connected. Assuming this to be true, since the complement of every labeled graph is unique, and no two graphs can have the same complement, the function associating to every disconnected graph its complement would be an injection between the set of disconnected graphs and the set of connected graphs on 10 vertices, implying that  $|\mathcal{D}| \leq |\mathcal{C}|$ .

Therefore, let  $D$  be an arbitrary disconnected graph,  $G$  be its complement, and  $u, v$  be two arbitrary vertices of  $D$ . We prove that we can always find a path from  $u$  to  $v$  in  $G$ . We can distinguish two cases:

- if  $(u, v)$  is not an edge in  $D$ , then  $u$  and  $v$  will be linked by an edge in a complement, so in this case we are done;
- if  $(u, v)$  is an edge in  $D$ , then since  $D$  is disconnected, there is another vertex  $y$  of  $D$  such that none of  $(u, y)$  and  $(v, y)$  are edges of  $D$ . But this implies that  $u - y - v$  is a path in  $G$ .

Thus, the complement of every disconnected graph is connected.

We are now left to prove the strict inequality. We know that the complement of a complement of a graph is the graph itself, so it is enough to find a connected graph whose complement is also connected. For this, let us pick the cycle on 10 vertices  $C_{10}$ . Note that  $C_{10}$  is connected, and its complement is connected as well, which finishes the proof.