Chapter 16

Geometric Discrepancy

Let H be a hypergraph with vertex set $V(H)$ and edge set $E(H)$, respectively. Let c be a coloring of the vertices with two colors $+1$ and -1, i.e., $c : V(H) \to \{+1, -1\}$. Given any edge E, let

$$c(E) = \sum_{x \in E} c(x) ,$$

and let

$$\operatorname{disc}(H, c) = \max_{E \in E(H)} |c(E)| .$$

The *discrepancy* of H is defined as

$$\operatorname{disc}(H) = \min_{c} \operatorname{disc}(H, c) ,$$

where the minimum is taken over all 2-colorings of the vertices. In other words, we want to find a 2-coloring which is as 'uniform' as possible, in the sense that every edge of H contains roughly the same number of points of color $+1$ and -1.

The theory of uniform distribution, especially discrepancy theory, has a long history, and it has several applications in number theory, geometry and numerical analysis. (See e.g. Beck-Chen (1987), Kuipers-Niederreiter (1974), Hlawka (1984), Beck and Sós (1994), Sós (1989).) It is not surprising that probabilistic and combinatorial methods play a central role in this field. However, it is intuitively less clear why harmonic analysis and integral geometry are instrumental in discrepancy theory. The aim of this chapter is to illustrate some of these techniques by concentrating on a few sample problems. In particular, we shall prove that the discrepancy of any hypergraph whose n vertices lie in the plane, and whose edges can be obtained by intersecting the vertex set with halfplanes, is at most $cn^{1/4}$, and that this bound is asymptotically tight (apart from the value of the constant c). These results are due to Matoušek (1994) and Alexander (1990), respectively, and improve some earlier estimates of Beck(1983b, 1991) by polylogarithmic factors.

We shall often use the following so-called *tail estimate* for the distribution of the sum of independent random variables.

Lemma 16.1 (Chernoff's inequality) *Let $\xi = \xi_1 + \xi_2 + \cdots + \xi_N$, where ξ_i are mutually independent random variables with $\Pr[\xi_i = +1] = \Pr[\xi_i = -1] = 1/2$. Then, for any $t > 0$,*

$$\Pr[\xi > t] < \exp\left(-\frac{t^2}{2N}\right).$$

Proof: Let c be a positive number to be fixed later. Then the expected value of $e^{c\xi_i}$,

$$\mathbf{E}[e^{c\xi_i}] = \frac{1}{2}e^c + \frac{1}{2}e^{-c} < e^{c^2/2}.$$

Since the ξ_i are mutually independent,

$$\mathbf{E}[e^{c\xi}] = \prod_{i=1}^{N} \mathbf{E}[e^{c\xi_i}] < e^{Nc^2/2}.$$

Thus,

$$\Pr[\xi > t] = \Pr[e^{c\xi} > e^{ct}] < \frac{\mathbf{E}[e^{c\xi}]}{e^{ct}} = e^{N\frac{c^2}{2} - ct}.$$

Setting $c = t/N$, we obtain the desired upper bound on $\Pr[\xi > t]$. □

Method of Floating Colors

Theorem 16.2 *Let H be hypergraph with n vertices and m hyperedges. Then*

$$\operatorname{disc}(H) \le \sqrt{2n \ln 2m}.$$

Proof: Take a random coloring c of the vertices, i.e., assign $+1$ or -1 to every vertex of H independently with probability $1/2$.

By Lemma 16.1, for any $E \in E(H)$,

$$\Pr[\,|c(E)| > t\,] < 2\exp\left(-\frac{t^2}{2|E|}\right) \le 2\exp\left(-\frac{t^2}{2n}\right).$$

Thus,

$$\begin{aligned}\Pr[\operatorname{disc}(H,c) > t] &\le \sum_{E \in E(H)} \Pr[\,|c(E)| > t\,] \\ &< 2m\exp\left(-\frac{t^2}{2n}\right) = 1,\end{aligned}$$

provided that $t = \sqrt{2n \ln 2m}$. Hence, there is a coloring c such that $\operatorname{disc}(H, c) \leq \sqrt{2n \ln 2m}$. □

A simple probabilistic construction shows that this bound in general cannot be much improved (see Exercise 16.1). However, if we assume that every vertex of H is contained only in a small number of hyperedges, then H can be colored more uniformly, i.e., with smaller discrepancy.

The number of edges $E \in E(H)$ containing a vertex x is called the *degree* of x in H, and is denoted by $d_H(x)$ (or by $d(x)$).

Theorem 16.3 (Beck-Fiala, 1981) *If every vertex of a hypergraph H has degree at most D, then* $\operatorname{disc}(H) \leq 2D$.

Proof: The idea of the proof is that we release the condition that a coloring can take only two values, $+1$ and -1. Instead of this, we shall consider a *quasi-coloring* $c : V(H) \to [-1, +1]$, whose values can lie anywhere in the closed interval $[-1, +1]$. We shall start with the trivial quasi-coloring $c \equiv 0$ having 0 discrepancy, and this will be gradually modified by setting some values to $+1$ or -1, until we obtain a true coloring. This procedure is widely known as the 'method of floating colors' (cf. Spencer (1987)).

Assume that at some stage of this construction we have a quasi-coloring c. If $c(x) = +1$ or -1, then we call this value *final,* and we shall not modify it later. If $-1 < c(x) < +1$, then x is said to have *floating colors.* A hyperedge $E \in E(H)$ is called *active*, if more than D of its points have floating colors. Suppose further that all active edges have 0 discrepancy, i.e.,

$$c(E) = \sum_{x \in E} c(x) = 0 \quad \text{for every active } E.$$

Suppose that not all edges of H are inactive, and let $E_1, E_2, \ldots, E_m$ denote the active edges. Renumber the vertices so that $x_1, x_2, \ldots, x_p$ have floating colors and the values of c are final on $x_{p+1}, \ldots, x_n$. Since every active edge has more than D points with floating colors, and every point belongs to at most D edges, we have $m < p$.

In the next step we shall modify the quasi-coloring c, as follows. Consider the system of m equations with p variables $(y_1, y_2, \ldots, y_p)$:

$$\sum_{\substack{x_j \in E_i \\ 1 \leq j \leq p}} y_j + \sum_{\substack{x_j \in E_i \\ p < j \leq n}} c(x_j) = 0 \quad (1 \leq i \leq m)\,.$$

The locus of points $(y_1, y_2, \ldots, y_p)$ satisfying all of these equations is an at least $(p - m) \geq 1$-dimensional flat in $\mathbb{R}^p$ having nonempty intersection with the cube $C^p = [-1, +1]^p$.