1 Detailed description of the problem

A graph $G = (V, E)$ is a pair consisting of a finite set of vertices $V$ and a set of edges $E$, such that each edge is an unordered pair of two distinct vertices in $V$. A drawing of a graph (or a topological graph) is a representation of the graph in the plane (or a higher genus surface) such that the vertices are represented by distinct points and the edges by (possibly crossing) simple continuous curves connecting the corresponding point pairs and not passing through any other point representing a vertex. Moreover, we assume that edges do not touch and that there are only finitely many edge crossings. We define the crossing number $\text{cr}(G)$ of a graph $G$ to be the smallest number of crossings in a drawing of $G$. We call two edges sharing a vertex adjacent. As a warm up exercise we offer a proof of the following fact:

**Fact 1.1** In a drawing of $G$ minimizing the number of edge crossings (i.e. a drawing witnessing $\text{cr}(G)$) adjacent edges do not cross. Moreover, in this drawing any pair of edges meet at most once and if they meet they either share a common vertex or they cross.

Other popular definition of crossing numbers are pair crossing number $\text{pcr}(G)$, which is the smallest number of pairs of edges crossing in a drawing of $G$ and odd crossing number $\text{ocr}(G)$, the smallest number of pairs of edges crossing oddly (odd pairs) in a drawing of $G$. By definition we have

$$\text{ocr}(G) \leq \text{pcr}(G) \leq \text{cr}(G)$$

It had been open for many years to decide whether $\text{ocr}(G) = \text{pcr}(G)$ for all graphs $G$ until finally in 2005 it was shown that there is an infinite family of graphs with $\text{ocr}(G) < 0.867 \cdot \text{pcr}(G)$ [5]. Later Tóth improved this result by giving a family of graphs with $\text{ocr}(G) < 0.855 \cdot \text{pcr}(G)$ [7].

However, to the best of my knowledge the question whether $\text{pcr}(G) = \text{cr}(G)$ holds for all graphs $G$ remains so far unsolved in general. In other words, we do not know whether there exists a graph that we can draw in the plane such that the number of crossing pairs of edges is less than its crossing number. A way how to attack this problem was recently proposed in [2]. Their method can be described as follows. Instead of studying the crossing numbers in general setting, we focus on a specific type of drawings, which are called monotone drawings. In a monotone drawing of a graph we represent each edge by an $x$-monotone curve. Moreover, we focus only on monotone drawings of graphs whose vertices are totally ordered and we require that in a drawing the $x$-coordinates of the points representing vertices respect this order i.e. the $x$-coordinate of the vertex $u$ is smaller than the $x$-coordinate of the vertex $v$ if and only if $u$ precedes $v$ in the given order.
We define monotone variants of crossing numbers of a graph $G$ whose vertices has been ordered by minimizing the number of crossings or crossing pairs of edges over all monotone drawings of $G$ respecting the given order of the vertices. Thus, we define *monotone crossing number* $\text{mon-cr}(G)$ of $G$ to be the smallest number of crossings in a monotone drawing of $G$ respecting the given order of its vertices. Then *monotone pair crossing number* $\text{mon-pcr}(G)$ is the smallest number of pairs of edges crossing in a monotone drawing of $G$ respecting the given order of its vertices and finally *monotone odd crossing number* $\text{mon-ocr}(G)$ is the smallest number of pairs of edges crossing oddly in a monotone drawing of $G$ respecting the given order of its vertices. By definition we have again

\[
\text{mon-ocr}(G) \leq \text{mon-pcr}(G) \leq \text{mon-cr}(G)
\]

In the upcoming journal version of [2] it was proved that a separation of two variants of monotone crossing numbers for ordered graphs implies a separation of the corresponding variants of crossing numbers in the general setting. Hence, to show that there exists a graph $G$ for which $\text{pcr}(G) < \text{cr}(G)$, it is enough to exhibit a graph $G'$ with ordered vertices for which $\text{mon-pcr}(G') < \text{mon-cr}(G')$.

In [3], it was shown that there is an infinite family of graphs such that for every graph in the family $\text{mon-ocr}(G) < \text{mon-pcr}(G)$ giving an alternative proof of the fact that there exists an infinite family of graphs such that $\text{ocr}(G) < \text{pcr}(G)$ for all graphs $G$ in this family.

2 Suggested stuff to work on

The ultimate goal would be to decide the following question:

**Question 2.1** Does $\text{mon-pcr}(G) = \text{mon-cr}(G)$ holds for all graphs $G$ with ordered set of vertices ?

A less challenging, but probably still not easy problem is the following:

**Question 2.2** Does $\text{mon-pcr}(G) = \text{mon-cr}(G)$ holds for all graph $G$ with ordered set of vertices, if we consider only the drawings in which every pair of edges cross at most twice ?

Other notions of crossing numbers were defined in [4] (see also [1, Section 9.4], or [6]) and similar questions can be also asked for these variants.

On a way towards resolving the above problems, there are many options how to proceed depending on our personal conjecture. Probably the most natural way is to start by looking at the properties of potential counter-examples to “YES” answer of the above questions minimizing certain quantity (e.g. number of vertices, edges, edge crossings). In this way we can introduce certain “forbidden configurations” that cannot appear in such a counter-example. If we are lucky and at least one element of our list of forbidden configuration is always unavoidable in every considered drawing of a graph we are done. Otherwise, we at least know something more about a potential counter-example. As a last resort to prove a separation there is a possibility to employ a computer-driven search for a graph witnessing a separation of different crossing numbers.

References


