Graph Theory: Problem set 9 Hints/Solutions

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1. Let G be a graph such that $\chi(G - x - y) = \chi(G) - 2$ for all pairs x, y of distinct vertices. Prove that G is complete.

Hint: Prove it by contradiction.

2. A graph is k-critical if $\chi(G) = k$ and $\chi(G - e) < k$ for any edge $e \in E(G)$. If G is k-critical, the show that G does not contain a cut set consisting of pairwise adjacent vertices.

Solution: Let S be a cutset. Let $H_1, .., H_t$ be the components of $G - S$. Since each $H_i \cup S$ is a proper subgraph, $H_i \cup S$ is $k - 1$ colorable. If each $H_i \cup S$ has a proper $k - 1$ coloring and S is a clique, then one can permute the colors such that G is $k - 1$ colorable which is a contradiction. Hence S is not a clique.

3. Show that every graph with chromatic number at least 4 contains a K_4-subdivision. (Hint: apply induction on the number of vertices)

Sketch Solution: Base case $n = 4$, then $G = K_4$. For the inductive step $n > 4$. Since $\chi(G) \geq 4$, we may let H be a 4-critical subgraph of G. Let S be the minimum-size set of vertices such that $H - S$ is disconnected. By the previous problem $|S| \neq 1$. If $|S| = 2$, then again by the previous problem, S must be an independent set. Hence on of the component of $G - S$ together with S must have chromatic number at least 4. Since this component has fewer edges, we can apply the induction hypothesis. Therefore we can assume $|S| \geq 3$ (i.e. G is 3-connected). Then one can find a cycle and apply the fan lemma to find a K_4 subdivision.

4. Construct a graph G with parallel edges (i.e. two vertices can have many edges between them), such that $\chi'(G) = 3\Delta(G)/2$.

Solution: Draw a "fat" triangle.

5. Given a graph G, let $L(G)$ denote the graph where the vertices of $L(G)$ are the edges of G, and two vertices in $L(G)$ are adjacent if and only if the corresponding edges share a vertex in G. Prove that the number of edges in $L(G)$ is $\sum\limits_{v \in V(G)} \binom{d(v)}{2}$.

Hint: Look at the "degree of an edge" and sum up over all edges.

6. $*$ If G is k-regular with a cut vertex, show that $\chi'(G) = \Delta(G) + 1$.